

# A DUALITY EXACT SEQUENCE FOR LEGENDRIAN CONTACT HOMOLOGY

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**ABSTRACT.** We establish a long exact sequence for Legendrian submanifolds  $L \subset P \times \mathbb{R}$ , where  $P$  is an exact symplectic manifold, which admit a Hamiltonian isotopy that displaces the projection of  $L$  to  $P$  off of itself. In this sequence, the singular homology  $H_*$  maps to linearized contact cohomology  $CH^*$ , which maps to linearized contact homology  $CH_*$ , which maps to singular homology. In particular, the sequence implies a duality between  $\text{Ker}(CH_* \rightarrow H_*)$  and  $CH^*/\text{Im}(H_*)$ . Furthermore, this duality is compatible with Poincaré duality in  $L$  in the following sense: the Poincaré dual of a singular class which is the image of  $a \in CH_*$  maps to a class  $\alpha \in CH^*$  such that  $\alpha(a) = 1$ .

The exact sequence generalizes the duality for Legendrian knots in  $\mathbb{R}^3$  [26] and leads to a refinement of the Arnold Conjecture for double points of an exact Lagrangian admitting a Legendrian lift with linearizable contact homology, first proved in [7].

## 1. INTRODUCTION

Legendrian contact homology, originally formulated in [2, 9], is a Floer-type invariant of Legendrian submanifolds that lies within Eliashberg, Givental, and Hofer's Symplectic Field Theory framework [10]. If  $(P, d\theta)$  is an exact symplectic  $2n$ -manifold with finite geometry at infinity, then Legendrian contact homology associates to a Legendrian submanifold  $L \subset (P \times \mathbb{R}, dz - \theta)$ , where  $z$  is a coordinate in the  $\mathbb{R}$ -factor, the “stable tame isomorphism” class of an associative differential graded algebra (DGA)  $(\mathcal{A}(L), \partial)$ . The algebra is freely generated by the Reeb chords of  $L$  — that is, integral curves of the Reeb field  $\partial_z$  that begin and end on  $L$  — and is graded using a Maslov index. The differential comes from counting holomorphic curves in the symplectization of  $(P \times \mathbb{R}, L)$ ; in the present case, this reduces to a count of holomorphic curves in  $P$  with boundary on the projection of  $L$ ; see [5, 8] and below.

In general, it is difficult to extract information directly from the Legendrian contact homology DGA. An important computational technique is Chekanov's linearization  $(Q(L), \partial_1)$  of  $(\mathcal{A}(L), \partial)$ . The linearization is defined when  $\mathcal{A}(L)$  admits an isomorphism that conjugates  $\partial$  to a differential  $\partial'$  which respects the word length filtration on  $\mathcal{A}(L)$ . In this case, the linearized homology is the  $E_1$ -term in the corresponding spectral sequence for computing the homology of  $\partial'$ . The linearized contact homology may depend on the choice of conjugating isomorphism, but the *set* of isomorphism classes of linearized contact homologies is

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TE was supported by the Alfred P. Sloan Foundation, by NSF-grant DMS-0505076, and by the Royal Swedish Academy of Sciences, the Knut and Alice Wallenberg Foundation. JE was partially supported by NSF grants DMS-0804820, DMS-0707509 and DMS-0244663.

An updated version of this article has appeared in the Duke Mathematical Journal, Volume 150 (2009), no. 1, 1–75.

invariant under deformations of  $L$ . Legendrian contact homology and its linearized version have turned out to be quite effective tools for providing obstructions to Legendrian isotopies [2, 6, 11, 18, 19, 22]. These results indicate the so-called “hard” properties of Legendrian embeddings.

The main result of this paper describes an important structural feature of linearized Legendrian contact homology. Say that a Legendrian submanifold  $L \subset P \times \mathbb{R}$  is *horizontally displaceable* if the projection of  $L$  to  $P$  can be completely displaced off of itself by a Hamiltonian isotopy. This condition always holds if  $P = \mathbb{R}^{2n}$  (or, more generally, if  $P = M \times \mathbb{C}$ ) or if  $L$  is a “local” submanifold that lies inside a Darboux chart.

**Theorem 1.1.** *Let  $L \subset P^{2n} \times \mathbb{R}$  be a closed, horizontally displaceable Legendrian submanifold whose projection to  $P$  has only transverse double points. If  $L$  is spin, then let  $\Lambda$  be  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , or  $\mathbb{Z}_m$ . Otherwise, let  $\Lambda$  be  $\mathbb{Z}_2$ .*

*If the contact homology of  $L$  is linearizable, then the linearized contact homology and cohomology and the singular homology of  $L$  with coefficients in  $\Lambda$  fit into a long exact sequence:*

$$(1.1) \quad \cdots \rightarrow H_{k+1}(L) \xrightarrow{\sigma_*} H^{n-k-1}(Q(L)) \rightarrow H_k(Q(L)) \xrightarrow{\rho_*} H_k(L) \rightarrow \cdots$$

*Furthermore, if  $\Lambda$  is a field, if  $\langle \cdot, \cdot \rangle$  is the pairing between the homology and cohomology of  $Q(L)$ , and if  $\bullet$  is the intersection pairing on homology of  $L$ , then for  $\gamma \in H_{n-k}(L)$  and  $\alpha \in H_k(Q(L))$*

$$\langle \sigma_*(\gamma), \alpha \rangle = \gamma \bullet \rho_*(\alpha).$$

A version of Theorem 1.1 was described for Legendrian 1-knots in  $J^1(\mathbb{R})$  in [26], where it was proved that off of a “fundamental class” of degree 1, the linearized Legendrian contact homology obeys a “Poincaré duality” between homology groups in degrees  $k$  and  $-k$ . Theorem 1.1 can also be interpreted as a Poincaré duality theorem for Legendrian contact homology, up to a fixed error term which depends only on the topology of the Legendrian submanifold. Any class in  $H_k(Q(L))$  that is in the kernel of  $\rho_*$  has a dual class in  $H^{n-k-1}(Q(L))$  determined up to the image of  $\sigma_*$ . In particular, there is a “duality” between  $\ker(\rho_*)$  and  $\text{coker}(\sigma_*)$ . We call the quotient  $H_k(Q(L))/\ker(\rho_*)$  the *manifold classes* of  $L$ . The last equation of Theorem 1.1 implies the following: if

$$V_{k;0} = \ker(\sigma_*) = \text{Im}(\rho_*) \subset H_k(L)$$

then

$$V_{n-k;0} = V_{k;0}^{\perp \bullet},$$

where  $V_{k;0}^{\perp \bullet}$  denotes the annihilator of  $V_{k;0}$  with respect to the intersection pairing. In particular, if  $\{\beta_1, \dots, \beta_r\}$  is a basis in  $V_{k;0}$  which is extended to a basis  $\{\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s\}$  and if  $\{\beta'_1, \dots, \beta'_r, \gamma'_1, \dots, \gamma'_s\}$  is the dual basis in  $H_{n-k}(L)$  under the intersection pairing then  $\{\gamma'_1, \dots, \gamma'_s\}$  is a basis in  $V_{n-k;0}$ . The map  $\rho_*$  induces an isomorphism between  $V_{k;0}$  and  $H_k(L)/\ker(\rho_*)$ . One may think of this isomorphism as a correspondence between pairs of Poincaré dual classes,  $(\beta_j, \beta'_j) \in H_k(L) \times H_{n-k}(L)$  and  $(\gamma'_l, \gamma_l) \in H_{n-k}(L) \times H_k(L)$ , and manifold classes in  $H_k(Q(L))/\ker(\rho_*)$  and  $H_{n-k}(Q(L))/\ker(\rho_*)$ , respectively. We prove in Theorem 5.5 that, over  $\mathbb{Z}_2$ , the fundamental class  $[L] \in H_n(L)$  is always in the image of  $\rho_*$ . In other words, in the pair of Poincaré dual classes  $([\text{point}], [L])$ , it is  $[L]$  which is hit by  $\rho_*$ .

For Poincaré dual pairs of homology classes other than this one, however, it is impossible to say *a priori* which class is in the image of  $\rho_*$ , as Example 5.10 shows.

An important application of Legendrian contact homology was to the Arnold conjecture for Legendrian submanifolds, which states that the number of Reeb chords of a generic Legendrian submanifold  $L \subset J^1(\mathbb{R}^n)$  is bounded from below by half the sum of Betti numbers of  $L$ . For Legendrian submanifolds with linearizable contact homology, this conjecture was proved in [7]. Theorem 1.1 gives the following refinement of that result.

**Theorem 1.2.** *Let  $L \subset P \times \mathbb{R}$  and  $\Lambda$  be as in Theorem 1.1, with the additional assumptions that the first Chern class of  $TP$  vanishes and that the Maslov number of  $L$  equals zero. Let  $c_m$  denote the number of Reeb chords of  $L$  of grading  $m$  and let  $b_k$  denote the  $k^{\text{th}}$  Betti number of  $L$  over  $\Lambda$ . If the Legendrian contact homology of  $L$  admits a linearization, then*

$$c_m + c_{n-m} \geq b_m$$

for  $0 \leq m \leq n$ .

The key to the proof of Theorem 1.1 is to study the Legendrian contact homology of the “two-copy” link  $2L$  consisting of  $L$  and another copy of  $L$  shifted high up in the  $z$ -direction. Holomorphic disks of  $2L$  admit a description in terms of holomorphic disks of  $L$  together with negative gradient flow lines of a Morse function on  $L$ . This description of holomorphic disks allows for a particularly nice characterization of the linearized contact homology of  $2L$  in terms of the linearized contact homology of  $L$  and its Morse homology. The observation that  $2L$  is isotopic to a link with no Reeb chord connecting different components then yields the exact sequence.

The paper is organized as follows. In Section 2, we set notation and review the basic definitions and constructions in Legendrian contact homology. The algebraic framework for the statement and proof of Theorem 1.1 is established in Section 3 using the description of holomorphic disks with boundary on  $2L$ . The analysis necessary for this description is carried out in Section 6. In Section 4, the duality theorem is established. Section 5 contains the proof of Theorem 1.2, and discusses, via examples, the relationship between the manifold classes in linearized contact homology and the singular homology of the Legendrian submanifold. In Appendix A, we establish some results related to gradings in contact homology.

*Acknowledgments:* The authors thank Frédéric Bourgeois for useful discussions which led to the current formulation of our main result.

## 2. BACKGROUND NOTIONS

This section sets terminology and gives a brief review of the basic ideas of Legendrian submanifold theory and Legendrian contact homology. See [5, 6, 7, 8] as well as Section 6 for details.

**2.1. The Lagrangian Projection into  $P$ .** Throughout the paper we let  $P$  denote an exact symplectic manifold with symplectic 2-form  $d\theta$ , primitive 1-form  $\theta$ , and finite geometry at infinity. We consider  $P \times \mathbb{R}$  as a contact manifold with contact 1-form  $dz - \theta$ , where  $z$  is a coordinate along the  $\mathbb{R}$ -factor. We use the projection  $\Pi_P : P \times \mathbb{R} \rightarrow P$  and call it the *Lagrangian projection*. If  $L \subset P \times \mathbb{R}$  is a Legendrian submanifold then  $\Pi_P(L) \subset P$  is a Lagrangian immersion with respect to the symplectic form  $d\theta$ .

As mentioned in the introduction, the Reeb vector field of  $dz - \theta$  is  $\partial_z$ . Consequently, if  $L$  is a Legendrian submanifold then there is a bijective correspondence between Reeb chords of  $L$  and double points of  $\Pi_P(L)$ . We will use “Reeb chord” and “double point” interchangeably, depending on context.

**2.2. Legendrian Contact Homology.** The Legendrian contact homology of a Legendrian submanifold  $L \subset P \times \mathbb{R}$  is defined to be the homology of a differential graded algebra (DGA) denoted by  $(\mathcal{A}(L), \partial)$ . In this paper, we will take the coefficients of  $\mathcal{A}(L)$  to be  $\Lambda$  where  $\Lambda$  is  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , or  $\mathbb{Z}_n$  if  $L$  is spin and  $\mathbb{Z}_2$  otherwise. In general, it is possible to define the DGA with coefficients in the group ring  $\Lambda[H_1(L)]$ .

**2.2.1. The Algebra.** Let  $L$  be a spin<sup>1</sup> Legendrian submanifold of  $(P \times \mathbb{R}, dz - \theta)$  such that all self-intersections of the Lagrangian projection of  $L$  are transverse double points; such a Legendrian will be called *chord generic*. (Note that if  $L \subset P \times \mathbb{R}$  is any Legendrian submanifold then there exists arbitrarily small Legendrian isotopies  $L_t$ ,  $0 \leq t \leq 1$  such that  $L_0 = L$  and such that  $L_1$  is chord generic.) Label the double points of the Lagrangian projection of  $L$  with the set  $\mathcal{Q} = \{q_1, \dots, q_N\}$ . Above each  $q_i$ , there are two points  $q_i^+$  and  $q_i^-$  in  $L$ , with  $q_i^+$  having the larger  $z$  coordinate.

Let  $Q(L)$  be the free  $\Lambda$ -module generated by the set of double points  $\mathcal{Q}$ , and let  $\mathcal{A}(L)$  be the free unital tensor algebra of  $Q(L)$ . This algebra should be considered to be a *based* algebra, i.e. the generating set  $\mathcal{Q}$  is part of the data. For simplicity, we will frequently suppress the  $L$  in the notation for  $Q(L)$  and  $\mathcal{A}(L)$ , and we will write elements of  $\mathcal{A}$  as sums of words in the elements of  $\mathcal{Q}$ .

**2.2.2. The Grading.** Before defining the grading, we need to make some preliminary definitions and choices. Let  $c_1(P)$  denote the first Chern class of  $TP$  equipped with an almost complex structure compatible with the symplectic form  $d\theta$ . Define the *greatest divisor*  $g(P)$  as follows: if  $c_1(P) = 0$ , then  $g(P) = 0$ ; otherwise, if  $c_1(P) \neq 0$ , then let  $g(P)$  be the largest positive integer such that  $c_1(P) = g(P)a$  for some  $a \in H^2(P; \mathbb{Z})$  such that  $g'a \neq 0$  if  $0 < g' < g(P)$ .

The set of complex trivializations of  $TP$  over a closed curve  $\gamma$  in  $P$  is a principal homogeneous space over  $\mathbb{Z}$ . In particular, given two trivializations  $Z$  and  $Z'$  of  $TP$  along  $\gamma$ , there is a well-defined distance  $d(Z, Z') \geq 0$  which is the absolute value of the class in  $\pi_1(GL(k, \mathbb{C})) \cong \mathbb{Z}$  determined by  $Z'$  if  $Z$  is considered as the reference framing.

**Definition 2.1.** Let  $g \in \mathbb{Z}$ ,  $g \geq 0$ . A  $\mathbb{Z}_g$ -framing of  $TP$  along a closed curve  $\gamma \subset P$  is an equivalence class of complex trivializations of  $TP$  along  $\gamma$ , where two trivializations  $Z_0$  and  $Z_1$  belong to the same equivalence class if  $d(Z_0, Z_1)$  is divisible by  $g$ .

In Appendix A, we show how choices of sections in certain frame bundles of  $TP$  over the 3-skeleton of some fixed triangulation of  $P$  induce a *loop  $\mathbb{Z}_g$ -framing of  $P$*  where  $g = g(P)$ , i.e., a  $\mathbb{Z}_g$ -framing of  $TP|_\gamma$  for any loop  $\gamma$  in  $P$ . We also show that such loop  $\mathbb{Z}_g$ -framings are unique up to an action of  $H^1(M; \mathbb{Z}_g)$ . If  $g = 0$ , then  $\mathbb{Z}_g$ -framings are ordinary framings and the framings are unique up to action of  $H^1(M; \mathbb{Z})$ . In the special case  $P = T^*M$ , it is straightforward to see that  $c_1(P) = 0$  and that there is a canonical loop framing; see Remark A.4. In what follows, we assume that a loop  $\mathbb{Z}_g$ -framing for  $P$  has been fixed.

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<sup>1</sup>This condition is unnecessary if we work with  $\mathbb{Z}_2$  coefficients.

If  $\gamma$  is a loop in  $L$ , then the tangent planes of  $L$  give a loop of Lagrangian subspaces of  $TP$  along  $\gamma$ . Using a trivialization  $Z$  representing the  $\mathbb{Z}_g$ -framing of  $TP|_\gamma$  (see Lemma A.3), we get a Maslov index

$$\mu(\gamma, Z) \in \mathbb{Z}$$

of the loop of tangent planes along  $\gamma$ . Since a change of complex trivialization by one unit changes the Maslov index by 2 units, this gives a homomorphism

$$H_1(L; \mathbb{Z}) \rightarrow \mathbb{Z}_{2g}.$$

A generator  $m(L)$  of the image of this homomorphism is called a *Maslov number of  $L$* .

We are now ready to define the grading. Assume that  $L \subset P \times \mathbb{R}$  is connected. For each of the double points in  $\mathcal{Q}$ , choose a *capping path*  $\gamma_i$  in  $L$  that runs from  $q_i^+$  to  $q_i^-$ . The Lagrangian projections of tangent planes to  $L$  along  $\gamma_i$  gives a bundle of Lagrangian subspaces over  $\gamma_i$  in  $TP$ . Pick a trivialization  $Z$  of  $TP$  over  $\gamma_i$  representing its  $\mathbb{Z}_g$ -framing. This gives a path  $\Gamma_i$  of Lagrangian subspaces in  $\mathbb{C}^n \approx \mathbb{R}^{2n}$ . Let  $\hat{\Gamma}_i$  be the result of closing the path  $\Gamma_i$  to a loop using a positive rotation (see [6]) along the complex angle between the endpoints of  $\Gamma_i$ . The *Conley Zehnder index*  $\nu(\gamma_i, Z)$  is the Maslov index of  $\hat{\Gamma}_i$ . Let

$$|q_i|_Z = \nu(\gamma_i, Z) - 1 \in \mathbb{Z}.$$

The grading  $|q_i|$  of  $q_i$  is

$$(2.1) \quad |q_i| = \pi(|q_i|_Z \bmod 2g) \in \mathbb{Z}_{lcm(2g, m(L))},$$

where  $\pi: \mathbb{Z}_{2g} \rightarrow \mathbb{Z}_{lcm(2g, m(L))}$  is the projection. Note that this grading is independent of the choice of capping path and of the choice of representative framing  $Z$ . It does, however, depend on the choice of loop  $\mathbb{Z}_g$ -framing.

The grading of a word in  $\mathcal{A}$  is the sum of the gradings of its constituent letters. This grading will be extended to the two-copy  $2L$  in Section 3.1.

**2.2.3. The Differential.** The differential on  $\mathcal{A}$  is a degree  $-1$  endomorphism that comes from counting rigid  $J$ -holomorphic disks in  $P$  with boundary on  $\Pi_P(L)$ . Here, we first give a general discussion of  $J$ -holomorphic disks and the moduli spaces they constitute. After that, we give brief definitions of properties of the almost complex structure  $J$  and of  $L$  that we will use to prove that the moduli spaces have certain regularity properties, see Lemmas 2.2 and 2.5. The proofs of these lemmas and more details on the definitions are found in Subsection 6.1. Finally, we define the differential.

A *marked disk*  $D_{m+1}$  is the unit disk in  $\mathbb{C}$  together with  $m+1$  marked points  $\{x_0, x_1, \dots, x_m\}$  in counter-clockwise order along its boundary. Let  $\partial \hat{D}_{m+1}$  be the boundary of  $D_{m+1}$  with the marked points removed. Over a double point  $q$  of  $\Pi_P(L)$  lie two points  $q^+$  and  $q^-$  in  $L$ ; let  $W^+$  be a neighborhood of  $q^+$  in  $L$ , and similarly for  $W^-$ . Given a continuous map  $u: (D_{m+1}, \partial D_{m+1}) \rightarrow (P, \Pi_P(L))$  with  $u(x_j) = q$ , say that  $u$  has a *positive puncture* (resp. *negative puncture*) at  $q$  if, as the boundary of  $D_{m+1}$  near  $x_j$  is traversed counter-clockwise, its image under  $u$  lies in  $\Pi_P(W^-)$  (resp.  $\Pi_P(W^+)$ ) before  $x_j$  and in  $\Pi_P(W^+)$  (resp.  $\Pi_P(W^-)$ ) after.

Now we can define the moduli spaces of holomorphic disks with boundary on  $L$ . Fix an almost complex structure  $J$  on  $P$  compatible with  $d\theta$ . For  $a, b_1, \dots, b_k \in \mathcal{Q}$  and  $A \in H_1(L)$ , define the moduli space  $\mathcal{M}_A(a; b_1, \dots, b_k)$ , to be the set of maps  $u: D_{m+1} \rightarrow P$  that are  $J$ -holomorphic, i.e. that satisfy  $du + J \circ du \circ i = 0$ , and so that:

- The boundary of the punctured disk is mapped to  $\Pi_P(L)$ ,
- The map  $u$  has a positive puncture at  $x_0$  with  $u(x_0) = a$ ,
- The map  $u$  has negative punctures at  $x_j$ ,  $j > 0$ , with  $u(x_j) = b_j$ , and
- The restriction  $u|_{\partial\widehat{D}_{m+1}}$  admits a continuous lift into  $L$  which, together with the capping paths  $-\gamma_a$  and  $\gamma_{b_j}$ , gives a loop that represents  $A$ .

These disks should be taken modulo holomorphic reparametrization. Note that

$$\mathcal{M}_A(a; b_1, \dots, b_k) = \bigcup_{\alpha} \mathcal{M}_A^{\alpha}(a; b_1, \dots, b_k),$$

where  $\alpha$  ranges over the homotopy classes represented by the disk maps. Consider a map of the boundary of a punctured disk corresponding to  $A \in H_1(L)$  and fix a homotopy class of disk maps  $\alpha$  with this boundary condition. Fix a trivialization  $TP$  along the boundary of the disk map which extends over the disk and fix representatives  $Z_a, Z_{b_1}, \dots, Z_{b_k}$  of the  $\mathbb{Z}_g$ -framings of the capping paths of  $a, b_1, \dots, b_k$  which agree with this trivialization at the double points. This gives a trivialization of  $TP$  along the closed-up loop representing  $A$ . Consider the loop of Lagrangian planes tangent to  $\Pi_P(L)$  along this closed-up loop and let  $\mu(A, \alpha)$  denote its Maslov index measured using the trivialization just described.

In order to show that moduli spaces of holomorphic disks are nice spaces we require the Legendrian  $L$  and the almost complex structure  $J$  to have certain properties. We use the following notation. As in [8], we say that an almost complex structure  $J$  on  $P$ , compatible with  $d\theta$ , is *adjusted to  $L$*  if there are coordinate neighborhoods near all double points of  $\Pi_P(L)$  where  $J$  looks like the standard complex structure on  $\mathbb{C}^n$ . We call such coordinates on a neighborhood of a double point *double point coordinates*. Given an almost complex structure adjusted to  $L$ , we say that  $L$  is *normalized at crossings* if its Lagrangian projection consists of linear subspaces near each double point. Coordinates on  $L$  near an endpoint of a Reeb chord that turn  $\Pi_P$  into a linear map into double point coordinates on  $P$  are called *Reeb chord coordinates*. Finally, a *symplectic neighborhood map of  $L$*  is a symplectic immersion  $\Phi$  from a neighborhood of the 0-section in  $T^*L$  which extends the Lagrangian immersion  $\Pi_P: L \rightarrow P$ . We say that an almost complex structure on  $P$  is *standard in a neighborhood of  $\Pi_P(L)$*  if it pulls back under some symplectic neighborhood map to an almost complex structure induced by a Riemannian metric  $g$  on  $L$ ; see Remark 6.1.

Let  $L$  be a chord generic Legendrian submanifold and let  $J_0$  be an almost complex structure adjusted to  $L$ . Assume that  $L$  is normalized at crossings and that  $J_0$  is standard in a neighborhood of  $\Pi_P(L)$ . Let  $N$  be a small closed regular neighborhood of  $\Pi_P(L)$  which is included in the region where  $J_0$  is standard. Write  $\mathcal{J}(N)$  for the space of almost complex structures, compatible with  $d\theta$ , which agrees with  $J_0$  in  $N$  and equip it with the  $C^2$ -topology. The following transversality result is closely related to Proposition 2.3 in [8].

**Lemma 2.2.** *Let  $L$  be a chord generic Legendrian submanifold. Then, after arbitrarily  $C^1$ -small deformation of  $L$ , there exists an almost complex structure  $J_0$  adjusted to  $L$ ,  $L$  is normalized at crossings, and  $J_0$  is standard in a neighborhood of  $\Pi_P(L)$ . Furthermore there exists a closed neighborhood  $N$  of  $\Pi_P(L)$  such that for an open dense set of almost complex structures  $J \in \mathcal{J}(N)$  the moduli space  $\mathcal{M}_A^{\alpha}(a; b_1, \dots, b_k)$  of  $J$ -holomorphic disks is*

a transversely cut out manifold of dimension  $d$ , where

$$d = |a|_{Z_a} - \sum_j |b_j|_{Z_{b_j}} + \mu(A, \alpha) - 1.$$

Lemma 2.2 is proved in Subsection 6.1.

We are now ready to discuss the moduli spaces involved in the definition of the differential. Fix an almost complex structure  $J$  so that Lemma 2.2 holds. (Later, in order to deal with the two-copy  $2L$  of  $L$ , see Subsection 3.1, we also need to fix a Morse function  $f$  so that Lemma 2.5 discussed below holds.)

If  $L$  is spin, the moduli spaces can be consistently oriented (see [7]). If  $\mathcal{M}_A(a; b_1, \dots, b_k)$  is zero dimensional, then by Gromov compactness it is compact and the algebraic count of points  $\#\mathcal{M}_A(a; b_1, \dots, b_k)$  makes sense. This count allows us to define the differential on generators as follows:

$$(2.2) \quad \partial a = \sum_{\dim(\mathcal{M}_A(a; b_1, \dots, b_k))=0} (\#\mathcal{M}_A(a; b_1, \dots, b_k)) b_1 \cdots b_k.$$

We then extend the differential to  $\mathcal{A}(L)$  via linearity and the Leibniz rule.

The following lemma combines the definition of  $\partial$  with Stokes' Theorem.

**Lemma 2.3.** *Let  $\ell(q_i)$  be the length of the Reeb chord lying above  $q_i$ . If  $u \in \mathcal{M}(a; b_1, \dots, b_k)$ , then:*

$$\ell(a) - \sum_k \ell(b_j) \geq C \text{Area}(u) > 0,$$

for some constant  $C > 0$ .

The central result of the theory is the following theorem.

**Theorem 2.4** ([5, 6, 7, 8]). *The differential  $\partial$  satisfies  $\partial^2 = 0$  and the “stable tame isomorphism class” (and hence the homology) of the DGA  $(\mathcal{A}, \partial)$  is invariant under Legendrian isotopy.*

See [2] for the definition of “stable tame isomorphism”; we will need only some straightforward consequences of the definition, not its precise formulation, in this paper.

**2.2.4. Another moduli space.** We will also need to consider moduli spaces of rigid holomorphic disks with boundary on  $L$  with exactly two positive punctures  $a_1, a_2$  and an arbitrary number of negative punctures. There are new transversality issues in this case, as the setup allows for multiple covers. In order to deal with such issues, we study closely related holomorphic disks, which in a sense give resolutions of multiple covers. More precisely, we take two copies  $L_0$  and  $L_1$  of  $L$  and let  $\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_l)$  denote the moduli space of holomorphic disks which are maps  $u : S \rightarrow P$ , where  $S$  is a strip  $\mathbb{R} \times [0, 1]$  with boundary punctures  $x_1, \dots, x_k$  appearing in order along  $\mathbb{R} \times \{0\}$  and  $y_1, \dots, y_l$  appearing in order along  $\mathbb{R} \times \{1\}$  with the following properties. The map  $u$  takes the puncture at  $-\infty$  to  $a_1$ , where  $a_1$  is a Reeb chord from  $L_1$  to  $L_0$ , the puncture at  $+\infty$  to  $a_2$  where  $a_2$  is a Reeb chord from  $L_0$  to  $L_1$ , the puncture  $x_j$  to  $b_j$ , where the  $b_j$ 's Reeb chords from  $L_0$  to  $L_0$ , and the puncture  $y_j$  to  $c_j$ , where the  $c_j$ 's are Reeb chords from  $L_1$  to  $L_1$ . It has positive punctures at  $\pm\infty$  and negative punctures elsewhere. It takes the boundary components in  $\mathbb{R} \times \{0\}$  to

$L_0$  and those in  $\mathbb{R} \times \{1\}$  to  $L_1$ , and  $A$  encodes the homology class of the boundary data as before. In complete analogy with the one-punctured case, we have

$$\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_l) = \bigcup_{\alpha} \mathcal{M}_A^{\alpha}(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_l),$$

where  $\alpha$  ranges over the homotopy classes of disk maps. Again, we choose a trivialization  $TP$  for the boundary of the disk which extends over the disk and is compatible with trivializations  $Z_c$  representing the  $\mathbb{Z}_g$ -framings of the capping paths of the Reeb chords of the disk.

In order to achieve transversality for  $\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_l)$ , we push  $L_1$  off of  $L = L_0$ . To state this more precisely, we require  $L$  to be normalized at crossings with respect to some almost complex structure  $J$  adjusted to  $L$ . Let  $f: L \rightarrow \mathbb{R}$  be a Morse function. We say that  $f$  is *admissible* if its critical points lie outside Reeb chord coordinate neighborhoods in  $L$  and if  $f$  is real analytic in Reeb chord coordinates near every Reeb chord endpoint.

Fix an admissible Morse function  $f: L \rightarrow \mathbb{R}$  which is sufficiently small so that the graph of  $df$  in  $T^*L$  lies in the domain of definition of a symplectic neighborhood map  $\Phi$  of  $L$ . Let  $L_1(f)$  be the image of the graph of  $df$  under  $\Phi$ . We write  $\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_l; f)$  for the moduli space of  $J$ -holomorphic curves with boundary on  $L_0 \cup L_1(f)$  and punctures as described above. We have the following result.

**Lemma 2.5.** *Let  $\hat{f}$  be a sufficiently small admissible Morse function and let  $k \geq 2$ . Then there are admissible Morse functions  $f$  arbitrarily  $C^k$ -close to  $\hat{f}$  such that if*

$$d = |a_1|_{Z_{a_1}} + |a_2|_{Z_{a_2}} - \sum_{j=1}^k |b_j|_{Z_{b_j}} - \sum_{j=1}^l |c_j|_{Z_{c_j}} - n + 1 + \mu(A, \alpha) \leq 0,$$

*then  $\mathcal{M}_A^{\alpha}(a_1, a_2; b_1, \dots, b_k, c_1, \dots, c_l)$  is a transversely cut out manifold of dimension  $d$ .*

Lemma 2.5 is proved in Subsection 6.1.

*Remark 2.6.* It is essential in the proof of Lemma 2.5 that the perturbations near one of the positive punctures are independent from those at the other. This is not true for multiply covered disks if the boundary of the disk lies on a single immersed Lagrangian  $L$ , but using two Lagrangian submanifolds resolves this problem: since the Reeb chord  $a_1$  starts on  $L_1$  and ends on  $L_0$  and the Reeb chord  $a_2$  starts at  $L_0$  and ends on  $L_1$ , there are no multiply covered disks in  $\mathcal{M}(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_r)$ . Furthermore, since both positive punctures map to mixed Reeb chords, “boundary bubbling” is not possible for topological reasons, i.e., broken disks which arises as limits of a sequence of disks in  $\mathcal{M}(a_1, a_2; b_1, \dots, b_k; c_1, \dots, c_r)$  must be joined at Reeb chords, not at boundary points; see [4, Definition 2.1].

*Remark 2.7.* Note that Lemma 2.5 gives one way of resolving the transversality problems arising from multiply covered disks. In order to compute the number of disks which would arise on the one-copy version, one would have to study the obstruction bundles over certain multiply covered disks on  $L$ .

**2.3. Linearization.** Legendrian contact homology is not an easy invariant to use, as it is a non-commutative algebra given by generators and relations. One important tool in extracting useful information from Legendrian contact homology is Chekanov's *linearized contact homology* [2]. To define it, break the differential  $\partial$  into components  $\partial = \sum_{r=0}^{\infty} \partial_r$ , where the image of  $\partial_0$  lies in the ground ring and  $\partial_r$  maps  $Q$  to  $Q^{\otimes r}$ . If  $\partial_0 = 0$  — that is, if there are no constant terms in the differential  $\partial$  — then the equation  $\partial^2 = 0$  implies that  $\partial_1^2 = 0$  as well, and hence that  $(Q, \partial_1)$  is a chain complex in its own right.

Even if the DGA does not satisfy  $\partial_0 = 0$ , it may be tame isomorphic to one that does; in this case, we call the DGA *good*. The existence of such an isomorphism is equivalent to the existence of an *augmentation* of the DGA  $(\mathcal{A}, \partial)$ , i.e. a graded algebra map  $\varepsilon$  from the algebra to the ground ring such that  $\varepsilon(1) = 1$  and  $\varepsilon \circ \partial = 0$ . To see why this is true, define an isomorphism  $\Phi^\varepsilon : \mathcal{A} \rightarrow \mathcal{A}$  on the generators of  $\mathcal{A}$  by:

$$(2.3) \quad \Phi^\varepsilon(q_i) = q_i + \varepsilon(q_i).$$

This map conjugates  $(\mathcal{A}, \partial)$  to a good DGA  $(\mathcal{A}, \partial^\varepsilon)$ . It is easy to check that  $\partial^\varepsilon$  may be computed from  $\partial$  by replacing  $q_i$  with  $q_i + \varepsilon(q_i)$  in the expression for the differential.

For any given augmentation  $\varepsilon$ , it is straightforward to compute the homology  $H_*(Q, \partial_1^\varepsilon)$ . The set  $\mathcal{H}(\mathcal{A}, \partial)$  of all such “linearized” homologies taken with respect to all augmentations of  $(\mathcal{A}, \partial)$  is an invariant of the stable tame isomorphism class of  $(\mathcal{A}, \partial)$  [2]. It is this set of linearized homologies for which we will prove duality.

Below, if  $\epsilon$  is an augmentation and if  $q \in Q$  is a generator then we say that  $q$  is *augmented* if  $\epsilon(q) \neq 0$ .

**2.4. Basic Examples.** Before proceeding to the proof of duality, let us look at a few basic examples in  $J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}$ . In these examples, we use the *front projection*  $\Pi_F : J^1(\mathbb{R}^n) \rightarrow J^0(\mathbb{R}^n) = \mathbb{R}^{n+1}$ . In the front projection, a Reeb chord occurs when two tangent planes to  $\Pi_F(L)$  over some point in  $\mathbb{R}^n$  are parallel.

In terms of the front projection, the Conley-Zehnder index of a capping path  $\gamma_i$  is calculated as follows. Let  $x_i \in \mathbb{R}^n$  be the projection of the double point  $q_i$  to  $\mathbb{R}^n$ . Around  $q_i^\pm$ , the front projection  $\Pi_F(L)$  is the graph of functions  $h_i^\pm$  with domain in a neighborhood of  $x_i$ . Define the difference function for  $q_i$  to be  $h_i = h_i^+ - h_i^-$ . Since the tangent planes to  $\Pi_F(L)$  are parallel at  $q_i^\pm$ , the difference function  $h_i$  has a critical point at  $x_i$  and, if  $L$  is chord generic, the Hessian  $d^2h_i$  is non-degenerate at  $x_i$ . For the proof of the following lemma, see [6].

**Lemma 2.8.** *If  $\gamma_i$  is a generic capping path for  $q_i$ , then*

$$(2.4) \quad \nu(\gamma_i) = D(\gamma_i) - U(\gamma_i) + I_{h_i}(x_i),$$

where  $D(\gamma_i)$  (resp.  $U(\gamma_i)$ ) is the number of cusps that  $\gamma_i$  traverses in the downward (resp. upward)  $z$ -direction and  $I_{h_i}(x_i)$  is the Morse index of  $h_i$  at  $x_i$ .

*Example 2.9* (The flying saucer). The front projection of the simplest Legendrian submanifold of  $\mathbb{R}^{2n+1}$  is shown in Figure 1. This Legendrian  $L$  is an  $n$ -sphere and  $\Pi_F(L)$  has exactly one double point  $c$ . The grading on this double point is  $|c| = n$ . Thus,  $\mathcal{A}(L)$  is the  $\mathbb{Z}$ -algebra generated by  $c$ . Moreover, for  $n \geq 2$ , one easily sees that  $\partial c = 0$  for degree reasons. Thus,



FIGURE 1. Front projection of the Flying Saucer in dimension 3, on the left, and 5, on the right.

the differential is good and the linearized contact homology is:

$$(2.5) \quad H_k(Q(L)) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

*Example 2.10.* Consider the two flying saucers in Figure 2. Connect them by a curve  $c(s)$  parametrized by  $s \in [-1, 1]$  that runs from the cusp edge of one flying saucer to the cusp edge of the other. Take a small neighborhood of  $c$  whose cross-sections are round balls whose radii decrease from  $s = -1$  to  $s = 0$ , and increase from  $s = 0$  to  $s = 1$ . Finally, introduce cusps along the sides of the neighborhood and join it smoothly to the two flying saucers. If  $n$  is even, the resulting Legendrian sphere  $L$  has the same classical invariants as the flying saucer. Note that this is Example 4.12 of [6].

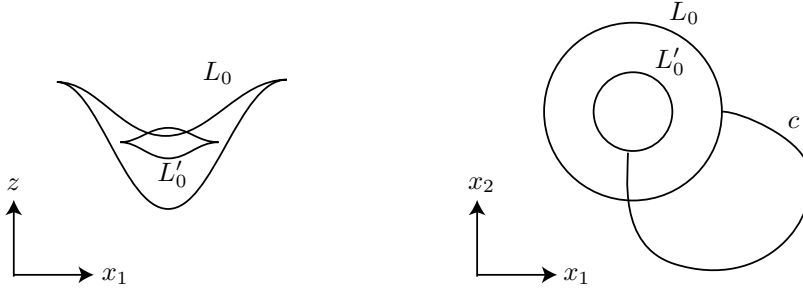


FIGURE 2. On the left side, the  $x_1z$ -slice of part of  $L_1$  is shown. To see this portion in  $\mathbb{R}^3$ , rotate the figure about its center axis. On the right side, we indicate the arc  $c$  connecting the two copies of  $L_0$ .

The connecting tube can be chosen such that there are six Reeb chords  $q_1, \dots, q_6$  that come from the two flying saucers and exactly one more Reeb chord  $q$  at the center of the connecting tube. The degrees of these chords are:

$$\begin{aligned} |q_1| &= |q_2| = |q_5| = n \\ |q_3| &= |q_6| = |q| = n - 1 \\ |q_4| &= 0. \end{aligned}$$

For  $n > 2$ , the fact that there are no chords of degree 1 implies that  $\mathcal{A}(L)$  must be a good algebra. Further, it is clear that  $H_0(Q(L)) \cong \mathbb{Z}$ . We will calculate the remainder of the linearized cohomology using Theorem 1.1 in Section 5.2.

*Remark 2.11.* In contrast to dimension three, where DGAs with  $\partial_0 = 0$  almost never arise directly from the geometry of a Legendrian knot, this is not infrequently the case in higher dimensions. In dimension three, however, the theory of the existence of augmentations has been well-studied (see [13, 14, 21, 24, 25]), while less work has been done in higher dimensions. One exception is Ng's use of augmentation ideals in his study of Knot Contact Homology [20], which is based on the technology of Legendrian contact homology.

### 3. ALGEBRAIC FRAMEWORK

In this section we present the algebraic setup for the linearized contact homology of the two-copy  $2L$  of a Legendrian submanifold  $L \subset P$ . The chain complex  $Q(2L)$  encodes the chain complexes for the linearized contact homology, the linearized contact cohomology, and the Morse homology of a function  $f$  on  $L$ . Similar setups for the two-copy of a Legendrian submanifold were used in [7] for the proof of double-point estimates of exact Lagrangians as well as in [26] for the proof of duality for Legendrian contact homology in three dimensions.

**3.1. The Two-Copy Algebra.** Let  $\phi_t: P \times \mathbb{R} \rightarrow P \times \mathbb{R}$  denote the time  $t$  Reeb flow. Note that  $\phi_t(p, t_0) = (p, t_0 + t)$ . The *two-copy*  $2L$  of a Legendrian submanifold  $L \subset P \times \mathbb{R}$  is the Legendrian link composed of  $L$  and  $\phi_s(L)$ , where  $s \gg 0$ . The Reeb chords of this link are degenerate: at every point of  $L$  which is not a Reeb chord endpoint, there starts a Reeb chord ending on  $\phi_s(L)$ , while at every Reeb chord endpoint of  $L$ , there start two such chords. This degenerate situation is similar to a Morse-Bott degeneration in Morse theory. We use the following perturbation of  $\phi_s(L)$  in order to get back into a generic situation. Let  $f$  be a  $C^1$ -small admissible Morse-Smale function on  $L$ . Perturb  $\phi_s(L)$  to  $df$  inside  $T^*\phi_s L$ , and then use the symplectic neighborhood map to transfer the result to  $P$ . Finally, lift the result back to  $P \times \mathbb{R}$ . We call this perturbed Legendrian submanifold  $\tilde{L}$ . Note that the  $z$ -coordinate of a point  $p$  in the new  $\tilde{L}$  differs by  $f(p)$  from the corresponding point in  $\phi_s(L)$ .

The generators of  $\mathcal{A}(2L)$  come from two sources. First, for every double point  $q$  of  $\Pi_P(L)$ , there are four double points of  $\Pi_P(2L)$ : two copies of the original double point of  $\Pi_P(L)$ , one denoted  $q^0$  in  $\Pi_P(L)$  and one denoted  $\tilde{q}^0$  in  $\Pi_P(\tilde{L})$ , and two intersections between  $\Pi_P(L)$  and  $\Pi_P(\tilde{L})$ . Second, each critical point of  $f$  gives a double point of  $\Pi_P(2L)$ . Thus, we can split the generators of the two-copy DGA into four types:

**Pure generators:** Double points  $q^0$  of  $\Pi_P(L)$  and their nearby counterparts  $\tilde{q}^0$  for  $\Pi_P(\tilde{L})$ ,

**Mixed  $q$  generators:** Intersections  $q^1$  between  $\Pi_P(L)$  and  $\Pi_P(\tilde{L})$  so that  $(q^1)^-$  lies near  $(q^0)^-$  and  $(q^1)^+$  lies near  $(\tilde{q}^0)^+$ ,

**Mixed  $p$  generators:** Intersections  $p^1$  between  $\Pi_P(L)$  and  $\Pi_P(\tilde{L})$  so that  $(p^1)^-$  lies near  $(q^0)^+$  and  $(p^1)^+$  lies near  $(\tilde{q}^0)^-$ , and

**Mixed Morse generators:** Double points  $c^1$  corresponding to critical points of  $f$ .

The first three types of generators are pictured schematically in Figure 3. For a small perturbation  $f$ , we have the following relationships between the lengths  $\ell$  of each type of

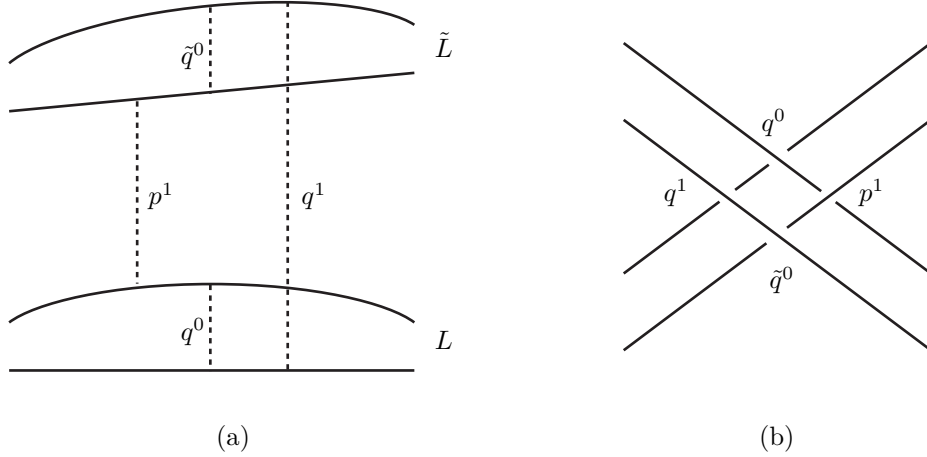


FIGURE 3. Three of the four types of generators of the two-copy algebra, as seen from (a) the front projection of  $2L$  and (b) the Lagrangian projection of  $2L$  for a knot in the standard contact  $\mathbb{R}^3$ .

chord:

$$\begin{aligned}
 \ell(q^1) &\approx s + \ell(q^0), \\
 \ell(c^1) &\approx s, \\
 \ell(p^1) &\approx s - \ell(q^0).
 \end{aligned}
 \tag{3.1}$$

To define the grading on  $\mathcal{A}(2L)$ , we follow the relative grading of [2, 19]. First, choose a basepoint  $p \in L$  away from the Reeb chords of  $2L$  and let  $\tilde{p}$  be its  $z$ -translate in  $\tilde{L}$ . Fix an identification of  $T_p L$  and  $T_{\tilde{p}} \tilde{L}$ . Next, choose capping paths  $\gamma_i$  for the Reeb chords of  $L$  that run through the marked point  $p$  and denote by  $\gamma_i^+$  and  $\gamma_i^-$  the paths  $p$  splits  $\gamma_i$  into, where  $\gamma_i^+$  ( $\gamma_i^-$ ) contains the starting point (ending point) of  $\gamma_i$ . Finally, choose capping paths  $\tilde{\gamma}_i$  for the double points on  $\tilde{L}$  to be  $z$ -translates of these (with small deformations near the endpoints when necessary). Given suitable trivializations of  $TL$  ( $T\tilde{L}$ ) along  $\gamma_i$  ( $\tilde{\gamma}_i$ ), we can then define loops  $\hat{\Gamma}_i^\pm$  of Lagrangian planes in  $\mathbb{R}^{2n}$  by concatenating  $\Gamma_i^+$  and  $\Gamma_i^-$  and closing up as before.

It is obvious that the  $q^0$  ( $\tilde{q}^0$ ) generators inherit their gradings from  $\mathcal{A}(L)$ . Using the concatenation property of the Conley-Zehnder index [23], it is easy to see that

$$|q_i^1| = |q_i^0|. \tag{3.2}$$

The loops  $\hat{\Gamma}_i^\pm$  for the  $p^1$  generators are simply the reverses of the loops for the  $q^1$  generators, so  $\nu(p_i^1) = n - \nu(q_i^1)$ . It follows that:

$$|p_i^1| = -|q_i^1| + n - 2. \tag{3.3}$$

For the gradings of the  $c^1$  generators, we work in local coordinates around the critical point in  $L$ , where we can consider  $L$  to be the 1-jet of the constant function and  $\tilde{L}$  to be the 1-jet of  $f$ . Choosing  $\gamma^+$  to be the reverse of a small perturbation of the translate of  $\gamma^-$ , we can

contract  $\Gamma$  into this neighborhood and use the proof of Lemma 2.8 in [6] to obtain:

$$(3.4) \quad |c_i^1| = \text{Index}_{c_i}(f) - 1.$$

In order to distinguish the differential  $\partial$  on  $\mathcal{A}(L)$  from the differential on  $\mathcal{A}(2L)$ , we will denote the latter by  $\widehat{\partial}$ . Furthermore, provided the Morse function  $f$  is sufficiently small, the algebras  $\mathcal{A}(L)$  and  $\mathcal{A}(\tilde{L})$  are canonically isomorphic DGAs. We will use this identification without further comment below.

**3.2. The Linearized Complex.** Assume that the contact homology of  $L \subset P \times \mathbb{R}$  is linearizable and let  $\varepsilon: \mathcal{A}(L) \rightarrow \Lambda$  be an augmentation. Define the algebra map  $\widehat{\varepsilon}: \mathcal{A}(2L) \rightarrow \Lambda$  on the generators by:

$$(3.5) \quad \widehat{\varepsilon}(x) = \begin{cases} \varepsilon(x) & x \text{ is a } q^0 \text{ or } \tilde{q}^0 \text{ generator,} \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\widehat{\varepsilon}$  is an augmentation of  $(\mathcal{A}(2L), \widehat{\partial})$ . To see this, first note that Lemma 2.3 and equation (3.1) imply that the differential of any  $q^0$  or  $\tilde{q}^0$  generator agrees with the original differential  $\partial$ . In particular,

$$\widehat{\varepsilon} \circ \widehat{\partial}(q^0) = \widehat{\varepsilon} \circ \widehat{\partial}(\tilde{q}^0) = 0.$$

Second, note that every term in the differential of a mixed Reeb chord must contain at least one more mixed chord. This implies:

$$\widehat{\varepsilon} \circ \widehat{\partial}(q^1) = \widehat{\varepsilon} \circ \widehat{\partial}(p^1) = \widehat{\varepsilon} \circ \widehat{\partial}(c^1) = 0,$$

and the augmentation property follows.

The techniques just used generalize to the following lemma.

**Lemma 3.1.** *If a holomorphic disk with boundary on  $2L$  has one positive puncture and its positive puncture maps to a mixed Reeb chord then it has exactly one negative puncture mapping to a mixed Reeb chord. If a holomorphic disk with boundary on  $2L$  has one positive puncture and its positive puncture maps to a pure Reeb chord then all its negative punctures map to pure Reeb chords as well.*

*Proof.* Immediate from Lemma 2.3 and Equation (3.1).  $\square$

Let us continue to explore the consequences of Lemma 2.3. Consider the chain complex  $Q(2L)$  of the linearized contact homology of  $2L$  with the differential  $\widehat{\partial}_1^{\widehat{\varepsilon}}$ . We decompose  $Q(2L)$  into four summands

$$Q(2L) = Q^1 \oplus C^1 \oplus P^1 \oplus Q^0;$$

where  $Q^1$  is generated by mixed  $q$  generators,  $C^1$  by mixed Morse generators,  $P^1$  by mixed  $p$ -generators, and  $Q^0$  by pure generators. Some components of the matrix of  $\widehat{\partial}_1^{\widehat{\varepsilon}}$  corresponding to this decomposition will turn the summands into chain complexes in their own right, while other components will become maps between these complexes. In the next section, we will reinterpret the complexes corresponding to the summands as the original linearized complex, its cochain complex, and the Morse-Witten complex of  $L$  with respect to the Morse function  $f$ . The maps between these complexes will give the exact sequence of Theorem 1.1.

*Remark 3.2.* For simpler notation below, we will suppress the augmentations from the notation for the differential writing simply  $\widehat{\partial}_1$  and  $\partial_1$  instead of  $\widehat{\partial}_1^\varepsilon$  and  $\partial_1^\varepsilon$ , respectively.

We next consider the decomposition of the differential corresponding to the direct sum decomposition above. Lemma 3.1 implies that  $Q^0 \subset Q(2L)$  is a subcomplex and that the component of  $\widehat{\partial}_1$  which maps  $Q^1 \oplus C^1 \oplus P^1$  to  $Q^0$  vanishes. Thus  $Q^1 \oplus C^1 \oplus P^1$  is a subcomplex as well. It is the subcomplex  $Q^1 \oplus C^1 \oplus P^1$  which carries the important information for the exact sequence. We therefore restrict attention to this subcomplex and leave  $Q^0$  aside.

Lemma 2.3 and Equation (3.1) imply that the linearized differential takes on the following lower triangular form with respect to the splitting  $Q^1 \oplus C^1 \oplus P^1$ :

$$(3.6) \quad \widehat{\partial}_1 = \begin{bmatrix} \widehat{\partial}_q & 0 & 0 \\ \rho & -\widehat{\partial}_c & 0 \\ \eta & \sigma & \widehat{\partial}_p \end{bmatrix}.$$

Since  $\widehat{\partial}_1^2 = 0$ , the diagonal entries in the matrix of  $\widehat{\partial}_1^2$  give

$$(3.7) \quad \widehat{\partial}_q^2 = 0, \quad \widehat{\partial}_c^2 = 0, \quad \text{and} \quad \widehat{\partial}_p^2 = 0.$$

The subdiagonal entries give

$$(3.8) \quad \rho \widehat{\partial}_q - \widehat{\partial}_c \rho = 0 \quad \text{and} \quad \widehat{\partial}_p \sigma - \sigma \widehat{\partial}_c = 0.$$

The last interesting entry in the matrix for  $\widehat{\partial}_1^2$  is

$$(3.9) \quad \eta \widehat{\partial}_q + \widehat{\partial}_p \eta + \sigma \rho = 0.$$

**Proposition 3.3.** *With notation as above, we have:*

- (1)  $(Q^1, \widehat{\partial}_q)$ ,  $(C^1, \widehat{\partial}_c)$ , and  $(P^1, \widehat{\partial}_p)$  are all chain complexes.
- (2) The maps  $\rho$  and  $\sigma$  are chain maps of degree  $-1$ .
- (3)  $\sigma_* \rho_* = 0$ , where  $\eta$  acts as a chain homotopy between  $\sigma \rho$  and the zero map.

*Proof.* This follows from (3.7), (3.8), and (3.9). □

Combining  $Q^1$  and  $C^1$  into  $QC^1 = Q^1 \oplus C^1$  gives another useful view of  $\widehat{\partial}_1$ . Let  $\widehat{\partial}_{qc} = -\widehat{\partial}_q - \rho + \widehat{\partial}_c$  and let  $H = \eta + \sigma$ . With respect to the splitting  $QC^1 \oplus P^1$ , then,  $\widehat{\partial}_1$  takes the following form:

$$(3.10) \quad \widehat{\partial}_1 = \begin{bmatrix} -\widehat{\partial}_{qc} & 0 \\ H & \widehat{\partial}_p \end{bmatrix}.$$

In parallel to Proposition 3.3, we obtain the following proposition.

**Proposition 3.4.** *With notation as above we have:*

- (1)  $(QC^1, \widehat{\partial}_{qc})$  is a chain complex.
- (2)  $H : (QC^1, \widehat{\partial}_{qc}) \rightarrow (P^1, \widehat{\partial}_p)$  is a chain map of degree  $-1$ .

Another way to look at the Proposition 3.4 is as follows:

**Corollary 3.5.** *The complex  $(QC^1 \oplus P^1, -\widehat{\partial}_{qc} + H + \widehat{\partial}_p)$  is the mapping cone of  $H$ . Further,  $QC^1$  itself is the mapping cone of  $\rho$ .*

As we shall see, the mapping cone of  $H$  is acyclic and hence  $H$  is an isomorphism. In fact, we will use this isomorphism to place the linearized contact cohomology inside the exact sequence.

**3.3. Identifying the Complexes in  $Q(2L)$ .** In this section, we identify the subcomplexes of  $Q(2L)$  discussed in the previous section with the original linearized chain complex of  $L$ , its cochain complex, and the Morse-Witten complex of  $L$ . The proofs of these identifications rest on an analytic theorem which describes all rigid holomorphic disks for a sufficiently small admissible perturbation function  $f$  satisfying certain technical conditions. We will describe that theorem here, but defer the detailed analytic treatment necessary for its proof to Section 6.

**3.3.1. Generalized Disks.** In order to state the analytic theorem for disks of the two-copy, we need to introduce terminology for objects built from holomorphic disks with boundary on  $L$  and flow lines of the Morse function  $f$  on  $L$  used to shift  $\phi_s(L)$  to  $\tilde{L}$ .

First, we make some assumptions. Let  $J$  be an almost complex structure adjusted to  $L$  and let  $L$  be normalized at double points. Assume that  $J$  is standard in a neighborhood of  $\Pi_P(L)$  with respect to a Riemannian metric  $g$  on  $L$ . Let  $f$  be an admissible Morse function which is round at critical points with respect to  $g$ ; see (a5) in Subsection 6.3. We call triples  $(f, g, J)$  with these properties *adjusted to  $L$* ; see Subsection 6.3. (The condition that the Morse function be round at critical points is inessential and made only in order to simplify some technicalities in proofs.)

Assume that the conclusions of Lemma 2.2 hold for  $J$ -holomorphic disks with boundary on  $L$ . Then as explained in Subsection 6.1, the compactified moduli space  $\overline{\mathcal{M}}$  of  $J$ -holomorphic disks with one positive puncture is a  $C^1$ -manifold with boundary with corners. Furthermore, as explained in Subsection 6.2, the corresponding compactified moduli space  $\overline{\mathcal{M}}^*$  of  $J$ -holomorphic disks with an additional marked point on the boundary comes equipped with a  $C^1$  evaluation map  $\text{ev}: \overline{\mathcal{M}}^* \rightarrow L$ . Further, assume that the admissible perturbing Morse function  $f$  and the Riemannian metric  $g$  on  $L$  are Morse-Smale and that, together with the almost complex structure  $J$ , they have the property that the stable and unstable manifolds of the critical points of  $f$ , defined using the  $g$ -gradient of  $f$ , are stratumwise transverse to the evaluation map  $\text{ev}: \overline{\mathcal{M}}^* \rightarrow L$ . For existence of adjusted triples  $(f, g, J)$  which satisfy these transversality conditions, see Lemma 6.5.

We define a *generalized disk* to be a pair  $(u, \gamma)$  consisting of a holomorphic disk  $u \in \mathcal{M} = \mathcal{M}_A(a; b_1, \dots, b_k)$  with boundary on  $L$  and a negative gradient flow line  $\gamma$  of the Morse function  $f$  beginning or ending at the boundary of  $u$ ; the other end of  $\gamma$  must lie at a critical point of  $f$ . We call the point on the boundary of  $u$  where the flow line  $\gamma$  begins or ends the *junction point* of  $(u, \gamma)$ . If the flow line starts at the junction point then  $(u, \gamma)$  has a *negative Morse puncture*; otherwise it has a *positive Morse puncture*. The *formal dimension*  $\dim((u, \gamma))$  of a generalized disk  $(u, \gamma)$  as follows.

$$(3.11) \quad \dim((u, \gamma)) = \begin{cases} \dim \mathcal{M} + 1 + (I_f(p) - n), & p \text{ a positive Morse puncture,} \\ \dim \mathcal{M} + 1 - I_f(p), & p \text{ a negative Morse puncture,} \end{cases}$$

where  $I_f(p)$  is the index of the critical point  $p$  at the end of  $\gamma$  which is not the junction point.

Generalized disks correspond to intersections of a stable/unstable manifold of  $f$  with the evaluation map  $\text{ev}: \overline{\mathcal{M}}^* \rightarrow L$ . The transversality conditions of the triple  $(f, g, J)$  of the Morse function, the metric, and the almost complex structure discussed above imply that such intersections are transverse. In particular, it follows from Lemma 6.5 that generalized disks determined by  $(f, g, J)$  have the following three properties: there are no generalized disks of formal dimension  $< 0$ , every generalized disk of dimension 0 is transversely cut out by its defining equation or *rigid*, and the set of rigid generalized disks is finite. (The third property is a consequence of the compactness of the moduli space of holomorphic disks and the transversality of the evaluation map). We say that triples  $(f, g, J)$  with these properties and for which rigid generalized disks satisfy certain pairwise transversality conditions as in Condition (g3) of Lemma 6.5, are *generic with respect to rigid generalized disks*.

We define a *lifted disk* to be a holomorphic disk  $u$  in the space  $\mathcal{M}_A(a; b_1, \dots, b_k)$  or  $\mathcal{M}_A(a_1, a_2; b_1, \dots, b_k, c_1, \dots, c_l)$  together with the following data: If  $u \in \mathcal{M}_A(a; b_1, \dots, b_k)$  then choose a puncture  $x_j$ ,  $j > 0$ , of the marked disk  $D_{m+1}$ . This puncture and  $x_0$  (which maps to the positive puncture  $a$ ) split  $\partial D_{m+1}$  into two components; assign the label  $L$  to one component and  $\tilde{L}$  to the other. If  $u \in \mathcal{M}_A(a_1, a_2; b_1, \dots, b_k, c_1, \dots, c_l)$ , note first that the projection of  $\tilde{L}$  into  $P$  agrees with the projection of  $L_1(f)$ , then we assign the label  $L$  to  $\mathbb{R} \times \{0\}$  and  $\tilde{L}$  to  $\mathbb{R} \times \{1\}$ .

Finally, we define a *lifted generalized disk*. Let  $(u, \gamma)$  be a generalized disk. The Morse flow line  $\gamma$  in  $L$  and its orientation reversed  $z$ -translate  $\tilde{\gamma}$  in  $\tilde{L}$  are two oriented curves, one which is oriented away from the junction point and one which is oriented toward it. The positive puncture and the junction point subdivide the boundary of the domain of  $u$  into two parts. One of these parts is oriented toward the junction point and the other one away from it. If the curve  $\gamma$  is oriented away from the junction point then we assign the component  $L$  to the part of the boundary of  $u$  which is oriented toward the junction point and  $\tilde{L}$  to the other part. If the curve  $\gamma$  is oriented toward the junction point then we assign the component  $L$  to the part of the boundary of  $u$  which is oriented away from the junction point and  $\tilde{L}$  to the other part.

We note that lifted disks, lifted generalized disks, and negative flow lines of  $f$  give rise to continuous maps from the boundary of a punctured disk to  $2L$  in a natural way. We call such maps *lifted boundary maps*.

We now have all of the objects we need to state the main analytic theorem:

**Theorem 3.6.** *There exist an admissible Morse function  $f: L \rightarrow \mathbb{R}$ , a Riemannian metric  $g$  on  $L$ , and an almost complex structure  $J$  on  $P$  with the following properties.*

- (i) *The pair  $(f, g)$  is Morse-Smale.*
- (ii) *The triple  $(f, g, J)$  is generic with respect to rigid generalized disks.*
- (iii) *Lemma 2.5 holds for  $J$ -holomorphic disks with boundary on  $L_0 \cup L_1(f)$ .*
- (iv) *All moduli spaces of  $J$ -holomorphic disks with one positive puncture and with boundary on  $2L = L \cup \tilde{L}$  of dimension  $\leq 0$  are transversely cut out. (Here, as above  $\tilde{L}$  is the push off of the Reeb flow image  $\phi_s(L)$  of  $L$  along  $df$ .)*

Furthermore, there is a bijective correspondence between the set  $X$  of rigid holomorphic disks with boundary on  $2L$  and the union  $Y$  of two (disjoint) copies of the set of rigid disks with boundary on  $L$ , rigid lifted disks determined by  $(f, g, J)$ , and rigid lifted generalized disks determined by  $(f, g, J)$ . This correspondence is such that the continuous lift of the boundary

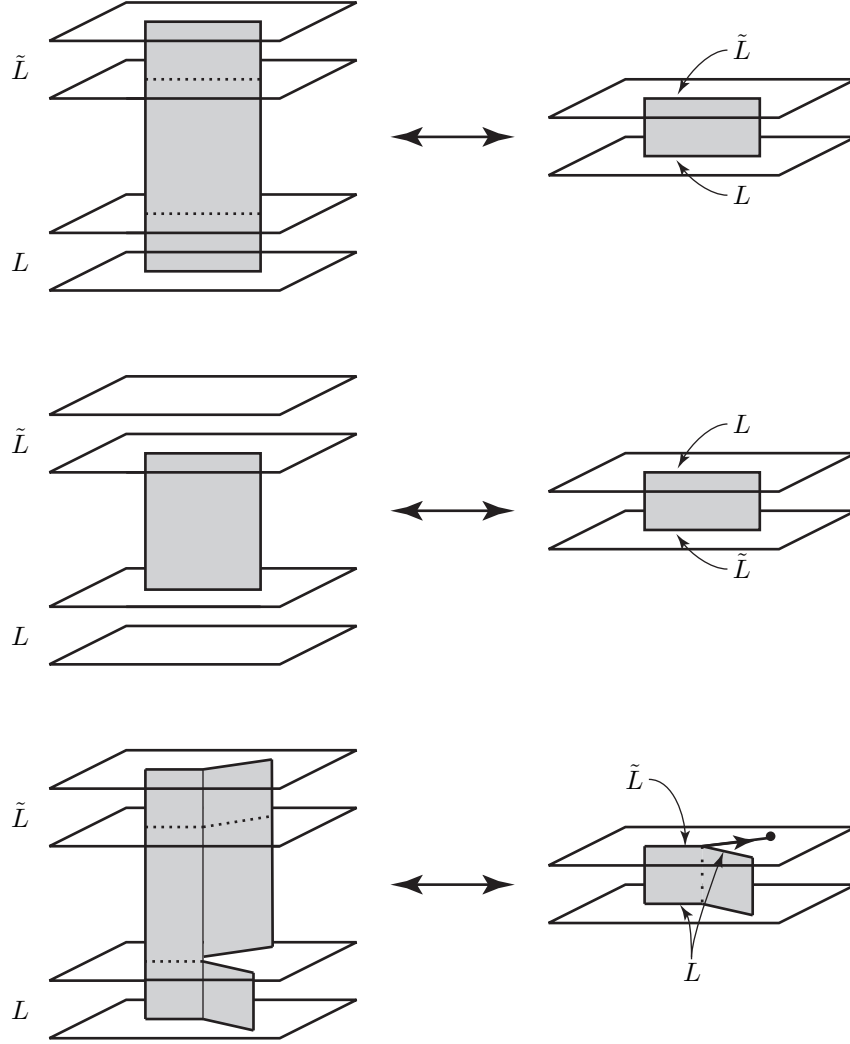


FIGURE 4. Schematic pictures of disks in the front diagrams of  $2L$  (on the left side) and  $L$  (on the right side) for parts (2) and (3) of the correspondence of Theorem 3.6. The top two diagrams illustrate part (2), while the bottom illustrates part (3). The labels on the right side are the labels of the lifted (generalized) disk.

of a holomorphic disk with boundary on  $2L$  is contained in a small neighborhood of the lifted boundary map of the object to which it corresponds, and vice versa. In particular:

- (1) The set of disks in  $X$  without mixed punctures corresponds to the subset of  $Y$  consisting of the two copies of the set of disks with boundary on  $L$ : disks with boundary on  $L$  corresponds to one copy and disks with boundary on  $\tilde{L}$  to the other.
- (2) The set of disks in  $X$  with mixed punctures but without mixed Morse punctures corresponds to the subset of  $Y$  of lifted disks. See Figure 4.

- (3) *The set of disks in  $X$  with exactly one Morse puncture corresponds to the subset of  $Y$  of lifted generalized disks. See Figure 4.*
- (4) *The set of disks in  $X$  with two mixed Morse punctures corresponds to the subset of  $Y$  of rigid negative gradient flow lines of  $f$ .*

It follows that the contact homology differential on  $\mathcal{A}(2L)$  can be computed in terms of rigid disks, rigid lifted generalized disks, and rigid lifted generalized disks. This theorem will be proven in Section 6.

**3.3.2. Chain Complexes.** Theorem 3.6 allows us to prove the following correspondences between chain complexes.

**Proposition 3.7.** *With notation as established in Section 3.2, we have:*

- (1)  $(Q^1, \widehat{\partial}_q)$  *is isomorphic to the original linearized chain complex  $(Q(L), \partial_1)$  and*
- (2)  $(C^1, \widehat{\partial}_c)$  *is isomorphic to the Morse-Witten complex  $CM_*(L; f)$  with respect to the perturbing Morse function  $f$ , with degrees shifted so that  $C_k^1 \cong CM_{k+1}(L; f)$ .*

*Proof.* For both parts of the proposition, there is an obvious bijective correspondence between the generators of the underlying graded groups (up to the indicated shift in the second part). We will now show that the disks contributing to the differentials also correspond.

For  $(Q^1, \widehat{\partial}_q)$ , a disk  $u$  that contributes  $q_k^1$  to  $\widehat{\partial}_q q_j^1$  has one positive mixed  $q$  puncture  $q_j^1$ , one negative mixed  $q$  puncture  $q_k^1$ , and possibly other negative augmented pure  $q$  punctures. By Theorem 3.6, this corresponds to a lifted disk with a positive puncture at  $q_k$ , a negative puncture at  $q_j$ , and possibly other augmented negative punctures. The rigid disk underlying the lifted disk contributes  $q_k$  to  $\partial_1 q_j$  in  $Q(L)$ , as required. Conversely, exactly one of the two lifts of such a rigid disk yields a disk with mixed  $q$  punctures at  $q_j$  and  $q_k$  and appropriate signs at the punctures.

For  $(C^1, \widehat{\partial}_c)$ , a disk  $u$  that contributes  $c_k^1$  to  $\widehat{\partial}_c c_j^1$  has a positive mixed Morse puncture at  $c_j^1$  and a negative mixed Morse puncture at  $c_k^1$ . Lemma 3.1 and Theorem 3.6 imply that this disk has only these two mixed Morse punctures and corresponds to a flow line from the critical point  $c_j$  to  $c_k$ , as in the Morse-Witten differential. The converse is clear.  $\square$

We next relate  $(P^1, \widehat{\partial}_p)$  to the cochain complex of  $(Q^1, \widehat{\partial}_q)$ . Define a pairing  $\langle, \rangle_q$  on the generators of  $P_{n-k-2}^1 \otimes Q_k^1$  by

$$(3.12) \quad \langle p_i^1, q_j^1 \rangle_q = \delta_{ij}.$$

**Proposition 3.8.** *The complex  $(P^1, \widehat{\partial}_p)$  is isomorphic to the cochain complex of  $(Q^1, \widehat{\partial}_q)$  with respect to the pairing  $\langle, \rangle_q$ .*

*Proof.* We need to show that

$$(3.13) \quad \langle \widehat{\partial}_p p_j, q_k \rangle_q = \langle p_j, \widehat{\partial}_q q_k \rangle_q.$$

A disk contributing to the left-hand side of (3.13) has a mixed positive puncture at  $p_j$ , a mixed negative puncture at  $p_k$ , and possibly other augmented pure punctures. By Theorem 3.6, such a disk corresponds to a unique rigid lifted disk. Reversing assignment of components of  $2L$  in the lifted rigid disk, we obtain a lifted disk which contributes to the

right-hand side; see the top two drawings in Figure 4. Similarly, to each disk contributing to the right hand side, we find a unique disk contributing to the left hand side.  $\square$

**3.3.3. The Maps  $\rho_*$  and  $\sigma_*$ .** In order to interpret Poincaré duality in the Morse-Witten complex, we first recall that if  $f_0$  and  $f_1$  are Morse functions on  $L$  with Morse-Smale gradient flows, then there is a continuation homomorphism  $h : CM_*(L; f_0) \rightarrow CM_*(L; f_1)$ . To define this map, choose a generic path of functions  $f_s$ ,  $s \in \mathbb{R}$ , which agrees with  $f_0$  for  $s \leq 0$  and with  $f_1$  for  $s \geq 1$ . The map is then given by counting solutions of the differential equation  $\dot{\gamma}(t) = -\nabla f_t(\gamma(t))$ ,  $t \in \mathbb{R}$ , which are asymptotic to critical points of  $f_0$  as  $t \rightarrow -\infty$  and to critical points of  $f_1$  as  $t \rightarrow +\infty$ .

Poincaré duality for  $H_*(L)$  can then be realized on the chain level in the Morse-Witten complex as follows (see [17] or [27], for instance): there is an obvious correspondence

$$\begin{aligned} \Delta : CM_k(L; -f) &\rightarrow CM^{n-k}(L; f) \\ x &\mapsto \langle x, \cdot \rangle. \end{aligned}$$

It induces an isomorphism  $\Delta_* : HM_k(L; -f) \rightarrow HM^{n-k}(L; f)$  on homology which, when combined with the continuation map  $h : CM_k(L; f) \rightarrow CM_k(L; -f)$ , yields the Poincaré duality isomorphism  $\Delta_* h_*$ . We can thus define a Poincaré pairing on  $HM_k(L; f) \otimes HM_{n-k}(L; f)$  by the following pairing at the chain level:

$$(3.14) \quad x \bullet y = (\Delta \circ h(x))(y).$$

Using this interpretation of Poincaré duality, we get the following relationship between the maps  $\rho$  and  $\sigma$ .

**Proposition 3.9.** *The maps  $\rho_*$  and  $\sigma_*$  are adjoints in the following sense:*

$$x \bullet \rho_* q = \langle \sigma_* x, q \rangle_q.$$

Proposition 3.9 is proved at the end of this section. The proof uses the following lemma.

**Lemma 3.10.** *The maps  $\rho_*$  and  $\sigma_*$  are independent of the perturbation function  $f$ . That is, if  $f_0$  and  $f_1$  are two generic perturbations and  $h_* : HM_*(L; f_0) \rightarrow HM(L; f_1)$  is the continuation map, then:*

$$\rho_*^1 = h_* \rho_*^0 \quad \text{and} \quad \sigma_*^1 h_* = \sigma_*^0,$$

where  $\rho^i$  and  $\sigma^i$  are the maps corresponding to the perturbation function  $f_i$ .

Lemma 3.10 uses some results from Morse theory which we present before giving its proof. Similar Morse theoretic questions were considered in [15, 28]. Our approach here is a modification of that in [28].

Denote the singular chain complex of  $L$  by  $C_*(L)$  and the corresponding homology by  $H_*(L)$ . Define a map  $\Phi^f : CM_*(L; f) \rightarrow C_*(L)$  as follows. If  $x$  is a critical point of  $f$ , then  $\Phi^f(x)$  is the singular chain carried by the closure of the unstable manifold  $\overline{W^u(x)}$ .

**Lemma 3.11.** *The map  $\Phi^f$  is a chain map that is compatible with the continuation map  $h_* : HM_*(L; f_0) \rightarrow HM_*(L; f_1)$  in the sense that  $\Phi_*^{f_1} h_* = \Phi_*^{f_0}$ .*

*Proof.* By Lemma 4.2 of [28], the boundary of  $\overline{W^u(x)}$  consists of products of rigid flow lines from  $x$  to  $y$  with  $\overline{W^u(y)}$ . The first part of the lemma follows. The second part is similar to the proof of Lemma 4.8 of [28], to which we refer the reader for the necessary gluing and compactness results: let  $x_j$  be critical points of  $f_j$ ,  $j = 0, 1$ , such that the number of rigid solutions of  $\dot{\gamma}(t) = -\nabla f_t(\gamma(t))$  asymptotic to  $x_0$  as  $t \rightarrow -\infty$  and  $x_1$  as  $t \rightarrow +\infty$  equals  $m$ . To construct a chain whose boundary is the chain carried by the closure of the unstable manifold  $\overline{W^u(x_0)}$  (defined with respect to  $f_0$ ) and  $m$  times the chain carried by the closure of the unstable manifold  $\overline{W^u(x_1)}$  (defined with respect to  $f_1$ ), consider the closure of the set of points which lie on some solution curve of  $\dot{\gamma}(t) = -\nabla f_t(\gamma(t))$  which is asymptotic to  $x_0$  as  $t \rightarrow -\infty$ .  $\square$

*Remark 3.12.* For technical reasons, we assume from here on that  $C_*(L)$  is generated by a set of  $C^1$  chains that is the union of both the chains in the image of  $\Phi^f$  and the chains in the image of the evaluation maps of all moduli spaces of holomorphic disks with boundary on  $L$ , one marked point, and at most two positive punctures.<sup>2</sup> Further, we assume that the  $C^1$  chains are transverse to the closures of the stable manifolds  $\overline{W^s(x)}$ . That this chain complex exists follows from arguments similar to those in [15, §19].

The inverse map  $\Psi^f : C_*(L) \rightarrow CM_*(L; f)$  is defined using intersections between chains and the stable manifolds of the critical points as follows. Let:

$$(3.15) \quad \Psi^f(\chi) = \sum_x (\chi \bullet W^s(x))x,$$

where  $\chi$  is a chain,  $\bullet$  denotes the intersection product on chains, and where the sum ranges over all critical points  $x$  of  $f$ . Note that the intersection product on chains is well-defined by our assumptions on the chains generating  $C_*(L)$ . Using the same proof as for Lemma 3.11, though this time based on ideas from Section 4.2 of [28], we obtain the following.

**Lemma 3.13.** *The map  $\Psi^f$  is a chain map compatible with the continuation map.*

Since it is clear that  $\Psi^f \circ \Phi^f$  is the identity on  $CM_*(L; f)$ , and since the Morse and singular homologies are isomorphic, it follows that  $\Phi_*^f$  and  $\Psi_*^f$  are mutual inverses.

We can now prove that  $\rho_*$  and  $\sigma_*$  are independent of the perturbation.

*Proof of Lemma 3.10.* Let  $\mathcal{M}(q)$  be the moduli space of holomorphic disks with positive puncture at  $q$  and possibly negative punctures at augmented crossings. Let  $K_q$  be the image in  $C_*(L)$  of  $\overline{\mathcal{M}}(q)$  under the evaluation map. Theorem 3.6 and the discussion before it imply that  $\rho$  may be defined using lifted generalized disks, so we obtain:

$$\rho^i(q) = \Psi^i(K_q).$$

If  $\alpha \in H_*(Q^1, \widehat{\partial}_q)$ , then:

$$\begin{aligned} \rho_*^0 \alpha &= \Psi_*^0(K_\alpha) \\ &= h_* \Psi_*^1(K_\alpha) \\ &= h_* \rho_*^1 \alpha. \end{aligned}$$

The proof for  $\sigma_*$  is similar.  $\square$

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<sup>2</sup>See Subsection 2.2.3, above, and [5, §7.7].

Finally, we are ready to prove the original proposition.

*Proof of Proposition 3.9.* Denote by  $\rho^f$  and  $\sigma^{-f}$  the maps induced by the perturbations  $\pm f$ . By Lemma 3.10, it suffices to prove that, on the chain level,

$$(3.16) \quad \tilde{h}(y) \bullet \rho^f q = \langle \sigma^{-f} y, q \rangle_q$$

for any given  $y \in CM_k(L; -f)$ , where  $\tilde{h}$  is the continuation map from  $CM_k(L; -f) \rightarrow CM_k(L; f)$ .

Theorem 3.6 says that a contribution to the left-hand side is given by a lifted generalized disk with a mixed positive puncture at  $q$  and a negative Morse puncture at  $y$  (and possibly other negative augmented pure punctures), where the flow line  $\gamma$  in the generalized disk is a negative gradient flow line for  $f$ . By reversing the lifting, we obtain a lifted generalized disk with a mixed negative puncture at the  $p$  corresponding to  $q$  and a positive Morse puncture at  $y$ . This time, however, the flow line  $\gamma$  points in the opposite direction, and hence is a negative gradient flow line for  $-f$ . Thus, this disk contributes to  $\sigma^{-f} y$ , and hence to the right hand side. Therefore, there is a bijective correspondence between the disks that determine the left- and right-hand sides of Equation (3.16).  $\square$

#### 4. THE DUALITY SEQUENCE

In this section, we prove Theorem 1.1 in two steps. First, we establish an isomorphism between  $QC^1$  and  $P^1$  (using notation as in Section 3.2) for  $L \subset P \times \mathbb{R}$ . Second, we collect the information from the identifications of Section 3.3.

**4.1. The Duality Isomorphism.** Assume that  $L \subset P \times \mathbb{R}$  is a Legendrian submanifold with satisfying the assumptions of Theorem 1.1. The first step will be to show that  $H_*$  is an isomorphism between  $QC^1$  and  $P^1$ .

**Proposition 4.1.** *The map  $H_* : H_*(QC^1) \rightarrow H_*(P^1)$  is a degree  $-1$  isomorphism.*

*Proof.* By Corollary 3.5, it suffices to prove that the complex

$$(QC^1 \oplus P^1, -\widehat{\partial}_{qc} + H + \widehat{\partial}_p)$$

is acyclic. Since  $(Q(2L), \widehat{\partial}_1)$  splits as the direct sum of two subcomplexes  $Q^0$  and  $QC^1 \oplus P^1$ , it is clear that

$$(4.1) \quad H_*(Q(2L)) \cong H_*(Q^0) \oplus H_*(QC^1 \oplus P^1).$$

In order to prove that  $QC^1 \oplus P^1$  is acyclic, we show that the linearized contact homology of  $2L$  comes from  $Q^0$  only, as follows.

Modify the two-copy by a Legendrian isotopy that leaves the bottom component fixed and horizontally displaces the top component by the lift of a Hamiltonian isotopy in  $P$  of the projection of  $L$  off of itself. For any augmentation, the linearized contact homology of this shifted two-copy is clearly isomorphic to the homology of  $Q^0$  since all of the other components of  $Q(2L)$  are trivial. Since the set of linearized contact homologies is invariant under Legendrian isotopy, this implies that, for any augmentation, the linearized contact homology of  $2L$  comes from  $Q^0$  only, as desired.  $\square$

*Remark 4.2.* It should not be surprising that  $H$  turns out to be an isomorphism essential to the proof of duality. In finite-dimensional Morse theory, rigid gradient flow trees with two positive ends and one vertex define the Poincaré duality isomorphism (see [1], for example). Using Theorem 3.6, it is straightforward to see that the disks that define  $H$  — or at least  $\eta$  — correspond to disks with two positive punctures and no non-augmented negative punctures in  $L$ . Thus, the disks defining  $H$  are combinatorially similar to the gradient flow trees that give Poincaré duality.

**4.2. The Proof of Theorem 1.1.** The complex  $QC^1$  is the mapping cone of  $\rho$ . The corresponding long exact sequence is:

$$(4.2) \quad \cdots \rightarrow H_k(C^1) \xrightarrow{i_*} H_k(QC^1) \rightarrow H_k(Q^1) \xrightarrow{\rho_*} \cdots,$$

where  $i : C^1 \rightarrow QC^1$  is the natural inclusion. By Proposition 3.7(2), we have  $H_k(C^1) \cong H_{k+1}(L)$ . By Propositions 3.8 and 4.1, the middle term  $H_k(QC^1)$  is isomorphic to  $H^{n-k-1}(Q^1)$ . Further, we have  $H_*i_* = \sigma_*$ . Inserting these facts into the exact sequence above yields:

$$(4.3) \quad \cdots \rightarrow H_{k+1}(L) \xrightarrow{\sigma_*} H^{n-k-1}(Q^1) \rightarrow H_k(Q^1) \xrightarrow{\rho_*} \cdots.$$

Proposition 3.7(1) now finishes the proof of the first part of the duality theorem. The second part of the duality theorem follows directly from Proposition 3.9.

## 5. APPLICATIONS AND EXAMPLES

**5.1. Proof of Theorem 1.2.** Let  $L \subset P \times \mathbb{R}$  be a Legendrian submanifold with linearizable contact homology over a field  $\Lambda$ . To set notation, let:

$$\begin{aligned} b_k &= \dim H_k(L), \\ r_k &= \dim \operatorname{Im} \rho_* \subset H_k(L), \\ s_k &= \dim \operatorname{Im} \sigma_* \subset H^{n-k}(Q(L)). \end{aligned}$$

Note that  $r_k$  is at most the dimension of  $H_k(Q(L))$ , which in turn bounds below  $c_k$ , the number of Reeb chords of grading  $k$ . By the second part of Theorem 1.1, we obtain  $s_k = r_{n-k}$ , so:

$$b_k = r_k + s_k = r_k + r_{n-k} \leq c_k + c_{n-k}.$$

**5.2. Basic Examples.** We study the relation between the linearized contact homology and the Morse homology implied by Theorem 1.1 in several simple examples.

*Example 5.1* (The Flying Saucer, revisited). Recall the flying saucer of Example 2.9, whose linearized homology (and cohomology) is  $\mathbb{Z}$  in degree  $n$  and trivial otherwise. The relevant part of the duality exact sequence in Theorem 1.1 is

$$\cdots \rightarrow H^{-1}(Q(L)) \rightarrow H_n(Q(L)) \xrightarrow{\rho_*} H_n(L) \rightarrow H^0(Q(L)) \rightarrow \cdots.$$

This reduces to  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$ . Thus, we see that  $\rho_*$  gives an isomorphism between the degree  $n$  linearized homology and the group generated by the fundamental class  $[L]$  of  $L \cong S^n$ .

Since  $H_0(Q(L)) = 0$ , and hence the image of  $\rho_*$  in  $H_0(L)$  is trivial, it is clear that the Poincaré dual of  $[L]$  evaluates to 0 on the image of  $\rho_*$  in  $H_0(L)$ , as guaranteed by the

Theorem 1.1. In fact, as we can see from the following part of the exact sequence,  $H_0(L)$  is isomorphic to the dual of  $H_n(Q(L))$

$$\cdots \rightarrow H_0(Q(L)) \rightarrow H_0(L) \xrightarrow{\sigma^*} H^n(Q(L)) \rightarrow H_{-1}(Q(L)) \rightarrow \cdots.$$

In other words, we see that the Morse homology of  $L$  splits between the linearized homology and the linearized cohomology.

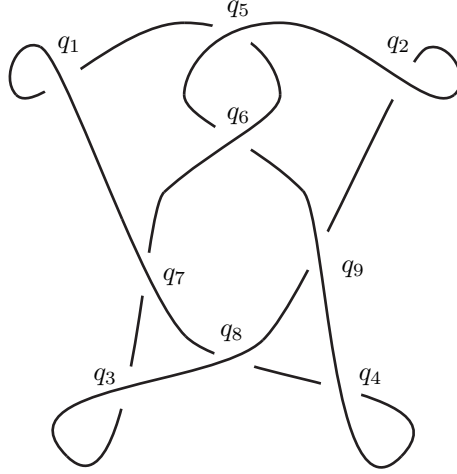


FIGURE 5. The knot  $L$ .

*Example 5.2* (Chekanov's Example in  $\mathbb{R}^3$ , reinterpreted). Consider the Legendrian knot  $L$  in  $\mathbb{R}^3$  whose Lagrangian projection is shown in Figure 5. Working over  $\mathbb{Z}_2$ , the algebra for  $L$  is  $\mathcal{A}(L) = \mathbb{Z}_2\langle q_1, \dots, q_9 \rangle$  with  $|q_i| = 1$  for  $i = 1, \dots, 4$ ,  $|q_5| = 2 = -|q_6|$ , and  $|q_i| = 0$  for  $i \geq 7$ . We have

$$\partial q_i = \begin{cases} 1 + q_7 + q_7 q_6 q_5 & i = 1, \\ 1 + q_9 + q_5 q_6 q_9 & i = 2, \\ 1 + q_8 q_7 & i = 3, \\ 1 + q_9 q_8 & i = 4, \\ 0 & i \geq 5. \end{cases}$$

This differential is not augmented, but there is a unique augmentation  $\varepsilon$  that sends  $q_7, q_8$ , and  $q_9$  to 1. The linearized differential is

$$(5.1) \quad \partial_1^\varepsilon q_i = \begin{cases} q_7 & i = 1, \\ q_9 & i = 2, \\ q_8 + q_7 & i = 3, \\ q_9 + q_8 & i = 4, \\ 0 & i \geq 5. \end{cases}$$

Thus, we have the following ranks for the linearized homology

$k$	$-2$	$-1$	$0$	$1$	$2$
$\dim H_k(Q(L))$	1	0	0	1	1

This computation agrees with the predictions of the duality theorem in [26]: off of a class in degree 1, the linearized homology is symmetric about degree 0.

The first interesting part of the duality exact sequence is

$$\cdots \rightarrow H_{-1}(L) \rightarrow H^2(Q(L)) \rightarrow H_{-2}(Q(L)) \rightarrow H_{-2}(L) \rightarrow \cdots .$$

This sequence reduces to  $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 0$  and shows that  $H^2(Q(L))$  and  $H_{-2}(Q(L))$  are isomorphic; this is the “duality” in Theorem 1.1.

The next interesting parts of the exact sequence are

$$\cdots \rightarrow H^{-1}(Q(L)) \rightarrow H_1(Q(L)) \rightarrow H_1(L) \rightarrow H^0(Q(L)) \rightarrow \cdots .$$

and

$$\cdots \rightarrow H_0(Q(L)) \rightarrow H_0(L) \rightarrow H^1(Q(L)) \rightarrow H_{-1}(Q(L)) \rightarrow \cdots .$$

As above, these sequences both reduce to  $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 0$ . As in the flying saucer example, we see that the homology of  $L \cong S^1$  is split between the linearized contact homology (in this case,  $H_1(L) \cong H_1(Q(L))$ ) and cohomology (in this case,  $H_0(L) \cong H^1(Q(L))$ ) in Poincaré dual pairs.

*Example 5.3 (Front Spinning).* Another source of examples in which the manifold classes follow an interesting pattern is the front spinning construction from [6]. Given a Legendrian submanifold  $L$  in  $\mathbb{R}^{2n+1}$ , we construct the *suspension*  $\Sigma(L)$  of  $L$  by “spinning the front of  $L$ ”. More specifically, suppose  $\phi: L \rightarrow \mathbb{R}^{2n+1}$  is a parametrization of  $L$ , and for  $p \in L$ , we write  $\phi(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p))$ . The front projection  $\Pi_F(L)$  of  $L$  is parametrized by  $\Pi_F \circ \phi(p) = (x_1(p), \dots, x_n(p), z(p))$ . We may assume that  $L$  has been translated so that the  $x_1$  coordinates of all points in  $\Pi_F(L)$  are positive. If we embed  $\mathbb{R}^{n+1}$  into  $\mathbb{R}^{n+2}$  via  $(x_1, \dots, x_n, z) \mapsto (x_0 = 0, x_1, \dots, x_n, z)$ , then  $\Pi_F(\Sigma L)$  is obtained by revolving  $\Pi_F(L) \subset \mathbb{R}^{n+1}$  around the subspace  $\{x_0 = x_1 = 0\}$  as in Figure 6. That is,

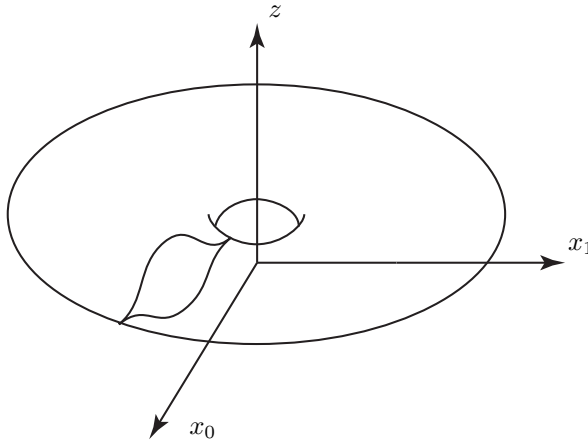


FIGURE 6. The front of  $\Sigma L$ .

we can parametrize  $\Pi_F(\Sigma L)$  by  $(x_1(p) \sin \theta, x_1(p) \cos \theta, x_2(p), \dots, x_n(p), z(p))$ , for  $\theta \in S^1$ .

Thus,  $\Pi_F(\Sigma L)$  is the front for a Legendrian embedding  $L \times S^1 \rightarrow \mathbb{R}^{2n+3}$ . We denote the corresponding Legendrian submanifold by  $\Sigma L$ .

We can derive the following facts about the contact homology DGA over  $\mathbb{Z}_2$  of  $\Sigma L$  from Proposition 4.17 of [6]: if  $\{q_1, \dots, q_n\}$  are the generators of  $\mathcal{A}(L)$ , then  $\mathcal{A}(\Sigma L)$  is stable tame isomorphic to an algebra generated by the set

$$\{q_1[\alpha], \dots, q_n[\alpha], \hat{q}_1[\beta], \dots, \hat{q}_n[\beta]\}_{\alpha=0,2,\beta=1,3}.$$

Let  $\Delta_\alpha : \mathcal{A}(L) \rightarrow \mathcal{A}(\Sigma L)$  take  $q_i$  to  $q_i[\alpha]$ , and similarly for  $\Delta_\beta$ . The gradings of the new generators are given by  $|q_i[\alpha]| = |q_i|$  and  $|\hat{q}_i[\beta]| = |q_i| + 1$ . Using the form of the differential in [6], we can see that if  $\varepsilon$  is an augmentation of  $(\mathcal{A}(L), \partial^L)$ , then the following formula defines an augmentation of  $(\mathcal{A}(\Sigma L), \partial^\Sigma)$

$$\begin{aligned} \varepsilon_\Sigma(q_i[\alpha]) &= \varepsilon(q_i), \\ \varepsilon_\Sigma(\hat{q}_i[\beta]) &= 0. \end{aligned}$$

We will use this form of augmentation from here on, though there might be others. Finally, the linearized differential takes the following form, where the augmentations have been suppressed from the notation

$$\begin{aligned} \partial_1^\Sigma q_i[\alpha] &= \Delta_\alpha(\partial_1^L q_i), \\ \partial_1^\Sigma \hat{q}_i[\beta] &= q_i[0] + q_i[2] + \Delta_\beta(\partial_1^L q_i). \end{aligned}$$

**Claim 5.4.** *The linearized homology of  $\Sigma L$  with respect to the augmentation  $\varepsilon_\Sigma$  may be computed using a Künneth-like formula*

$$H_*(Q(\Sigma L)) \simeq H_*(Q(L)) \otimes H_*(S^1).$$

*Proof.* To prove the claim, we change basis by replacing  $\hat{q}_i[3]$  by  $\hat{q}_i[1] + \hat{q}_i[3]$ . The subspace  $Q[1+3]$  spanned by these new basis elements forms a subcomplex that is isomorphic to the original  $(Q(L), \partial_1^L)$ , but with gradings shifted up by 1. Similarly, the subspaces  $Q[0]$  spanned by the  $q_i[0]$  and  $Q[2]$  spanned by the  $q_i[2]$  also form subcomplexes isomorphic to  $(Q(L), \partial_1^L)$ . Finally, the restriction of the differential to the subspace  $Q[1]$  yields no new cycles. Further, if  $z$  is a cycle in the original chain complex, then the differential of  $\Delta_1(z)$  identifies the cycles  $z[0]$  and  $z[2]$  in homology. Thus, the linearized homology of  $\Sigma L$  comes from  $Q[1+3]$  and  $Q[0]$ , each of which are copies of the original chain complex  $Q(L)$ , with degrees in  $Q[1+3]$  shifted up by one.  $\square$

If we apply the spinning construction  $k$  times to the  $n$ -dimensional “flying saucer” of Example 5.1, the claim above shows that the linearized homology corresponds to the homology of  $T^k$  with degrees shifted up by  $n$ . This is half of the homology of  $S^n \times T^k$ , and hence the linearized homology consists entirely of manifold classes.

For a more interesting example, apply the spinning construction  $k$  times to the Legendrian knot  $L$  in Example 5.2 to obtain a Legendrian  $T^{k+1} \subset \mathbb{R}^{2k+3}$ . Setting  $k = 2$  and using the calculation of the linearized homology in Example 5.2 and Claim 5.4, we easily obtain the following dimensions

$k$	-2	-1	0	1	2	3	4
$\dim H_k(Q(\Sigma^2 L))$	1	2	1	1	3	3	1

The duality theorem implies that the non-manifold classes are symmetric about  $\frac{3-1}{2} = 1$ , and we can see that every non-manifold class in  $H_k(Q(L))$  has been replaced by  $H_*(T^2)$ , shifted by  $k$ . The remaining classes are the manifold classes, corresponding to  $H_*(T^2)$  shifted up by one degree.

**5.3. Finding Manifold Classes.** As indicated by the examples above, it is sometimes possible to use symmetry arguments to predict the degree in which manifold classes appear. In particular, working over  $\mathbb{Z}_2$ , we can generalize the fact from [26] that for a Legendrian *knot*, there is always a manifold class in degree 1 as follows.

**Theorem 5.5.** *Suppose  $L \subset P \times \mathbb{R}$  satisfies the assumptions of Theorem 1.1. If the contact homology DGA of  $L$  is good, then  $\rho_*$  is trivial in degree 0 and onto in degree  $n$ . That is, the linearized homology of  $L$  (with respect to any augmentation) always has a  $\mathbb{Z}_2$  factor in degree  $n$  corresponding to the fundamental class of  $L$ .*

*Proof.* By the second part of Theorem 1.1, it suffices to prove that  $\rho_*$  is trivial in degree 0. To compute  $\rho$  in degree zero, we need to find lifted generalized disks  $(u, \gamma)$  in  $L$  with one negative Morse puncture and a positive mixed puncture at  $q$  with  $|q| = 0$ . The fact that this generalized disk is rigid implies, via Equation (3.11), that  $u$  belongs to a moduli space of dimension  $-1$ . That is,  $u$  must be a constant map to the double point  $q$  with one positive and one negative puncture. Further,  $q$  must be augmented, or else the generalized disk will not contribute to  $\rho$ .

There are two possibilities for the flow line  $\gamma$ . There are two sheets of  $\Pi_P(L)$  incident to  $q$ , and the flow line  $\gamma$  can start on either of these two sheets. In a suitably generic setup, the self-intersection points of  $\Pi_P(L)$  are disjoint from the stable manifolds of the critical points of  $f$  with index greater than 0, so the two flow lines starting at  $q$  both descend to a minimum of  $f$ . It follows that  $\rho(q)$  is either zero or the sum of two minima of  $f$ . Thus, given a homology class  $\alpha \in H_0(Q^1)$ , its image under  $\rho_*$  is the sum of an even number of minima. As  $L$  is connected, these minima are all homologous, so  $\rho_*\alpha = 0$ .  $\square$

*Remark 5.6.* Provided certain orientation conventions are employed, Theorem 5.5 holds for more general coefficients in the case that  $L$  is spin. More precisely, the two disks corresponding to the flow lines in the last argument of the proof should cancel with signs. The ordered punctures of these two disks are of the forms  $(q^1, c^1, q^0)$  and  $(q^1, \tilde{q}^0, c^1)$ , respectively. Choosing capping operators as in [7], the capping operator of  $\tilde{q}^0$  is identical to that of  $q^0$  and both have index 1. The capping operator of  $c^1$  has index 0 if  $n = \dim(L)$  is odd and index 1 if  $n$  is even. Thus, the disks cancel also with signs if  $n$  is even. If  $n$  is odd, they cancel with signs provided we redefine the augmentation  $\hat{\varepsilon}$  on the  $\tilde{q}^0$ -chords by declaring that  $\hat{\varepsilon}(\tilde{q}^0) = -\varepsilon(q)$  instead. Here  $q$  is the Reeb chord of  $L$  which corresponds to the Reeb chord  $\tilde{q}^0$  of  $\tilde{L}$ , and  $\varepsilon$  is the augmentation on  $\mathcal{A}(L)$ .

**Corollary 5.7.** *Let  $L$  be a Legendrian homology  $n$ -sphere satisfy the assumptions of Theorem 1.1. If the contact homology DGA of  $L$  is good, then the linearized homology of  $L$  (with respect to any augmentation) always has a  $\mathbb{Z}_2$  factor in degree  $n$  and, off of this factor, the remaining homology is symmetric about  $\frac{n-1}{2}$ . Said another way, if  $P(t)$  is the Poincaré-Chekanov polynomial of the linearized homology, then:*

$$P(t) - t^{n-1}P(t^{-1}) = (t^n - t^{-1}).$$

Theorem 5.5 and Corollary 5.7 can greatly ease computations, as can be seen in the next example; see also [16].

*Example 5.8* (Example 2.10, revisited). Recall Example 2.10, in which two flying saucers are attached by a tube. For  $n > 2$ , the algebra is good for degree reasons, with three generators in degree  $n$ , three in degree  $n - 1$ , and one in degree 0. The fact that the generator of degree 0 is isolated shows that  $\dim H_0(Q(L)) = 1$ , and Corollary 5.7 immediately implies that

$$P(t) = 1 + t^{n-1} + t^n.$$

In particular, the duality theorem forces nonzero differentials between degrees  $n$  and  $n - 1$ .

*Example 5.9* (A Non-Spun Torus). The previous example may be combined with Claim 5.4 to produce an example of a Legendrian  $S^1 \times S^n$  that is not spun from a Legendrian  $S^n$  for  $n > 1$ . Let  $L$  be the Legendrian  $n + 1$ -sphere constructed in the previous example, and let  $L'$  be standard  $n$ -dimensional flying saucer. Taking the connect sum  $L \# \Sigma L'$  as in [6] yields a Legendrian submanifold with four generators in degree  $n + 1$ , five generators in degree  $n$ , and one in degree 0. As before, the fact that the generator of degree 0 is isolated shows that  $\dim H_0(Q(L)) = 1$ , and Theorem 1.1 and 5.5 immediately imply that

$$P(t) = 1 + 2t^n + t^{n+1}.$$

This homology, however, is not of the form  $H_*(Q(L'')) \otimes H_*(S^1)$  near degree 0, so  $L \# \Sigma L'$  cannot be a spun submanifold.

The final example shows that Theorem 5.5 is the best that we can hope for in terms of pinning down the degrees of the manifold classes.

*Example 5.10* (Super-spun Products of Spheres). Construct a generic front diagram in  $\mathbb{R}^{p+k+1}$  as follows: let  $S^p \times S^k \hookrightarrow \mathbb{R}^{p+k+1}$  be the standard embedding, with  $S^p \times S^k = \partial(S^p \times B^{k+1})$ . Deform the embedding so that the  $S^k$  cross-sections are fronts for the flying saucer (see Example 2.9). Let  $F : S^p \rightarrow \mathbb{R}$  be a Morse function with one maximum and one minimum, and scale each cross section  $\{x\} \times S^k$  by  $1 + \epsilon F(x)$  for some small  $\epsilon > 0$ .

The Legendrian embedding  $L$  coming from this front has exactly two Reeb chords, one of degree  $k + p$  and the other of degree  $k$ . Suppose, for convenience, that  $p > k > 1$ , so that the algebra  $\mathcal{A}(L)$  is good for degree reasons. Direct calculation then shows that the linearized homology is  $\mathbb{Z}$  in degrees  $k + p$  and  $k$ , and zero otherwise. Theorem 1.1 shows that the homology class of degree  $k$  is a manifold class coming from  $H_k(S^p \times S^k)$ .

Reversing the roles of  $k$  and  $p$  yields another Legendrian embedding of  $S^p \times S^k$ , but this time with a linearized homology class in degree  $p$  is a manifold class that comes from  $H_p(S^p \times S^k)$ . Thus, it is impossible to determine a priori which classes in  $H_*(L)$  in degrees other than 0 or  $n$  will be manifold classes.

## 6. PROOF OF THE MAIN ANALYTIC THEOREM

This section is devoted to the proof of Theorem 3.6, which describes the rigid holomorphic disks with boundary on  $2L$  in terms of holomorphic disks with boundary on  $L$  and flow lines of a Morse function on  $L$ . We present the main steps of the proof in several subsections as follows: in Subsection 6.1, we endow compactified moduli spaces of holomorphic disks with boundary on  $L$  with a structure of a manifold with boundary with corners, and in

Subsection 6.2, we define evaluation maps on such compactified moduli spaces of disks with an additional marked point. In Subsection 6.3, we study notions of genericity for generalized disks and prove existence results for Morse functions and for almost complex structures with desired transversality properties.

In Subsections 6.4 and 6.5, we turn to the correspondence between rigid disks with boundary on  $2L$  and rigid lifted and lifted generalized disks on  $L$ . In particular, we show how holomorphic disks with boundary on  $2L$  converge to generalized disks as the perturbation of the second copy approaches zero and how generalized disks can be glued to holomorphic disks with boundary on  $2L$  for small enough separation, respectively. Finally, in Subsection 6.6, we prove Theorem 3.6.

**6.1. Compactified moduli spaces as manifolds with boundary with corners.** Moduli spaces of  $J$ -holomorphic disks with boundary on  $L$  admit natural compactifications consisting of broken holomorphic disks. In order to describe these compactified spaces, we first describe the smooth pieces out of which they are built. This description is provided by Lemma 2.2. We prove this lemma and Lemma 2.5 in Subsection 6.1.1 after recalling some properties of almost complex structures on cotangent bundles induced by Riemannian metrics. With these lemmas established, we show how to glue the smooth pieces together to form a manifold with boundary with corners. More precisely, in Subsection 6.1.2, we describe the broken disks that compactify the moduli space. In Subsections 6.1.3 and 6.1.4, we describe a basic tool for gluing and how to express the problem of gluing broken holomorphic disks in a language suitable for using this basic tool. In Subsection 6.1.5, we start our construction of coordinate charts by constructing approximately holomorphic disks near broken disks and in Subsection 6.1.6, we complete it. Finally, in Subsection 6.1.7, we show that the coordinate charts fit together to give the desired structure of a manifold with boundary with corners.

#### 6.1.1. Proofs of Lemmas 2.2 and 2.5.

*Remark 6.1.* Since we will use some properties of the almost complex structure on a cotangent bundle induced by a Riemannian metric, we include a short discussion of its definition. Let  $q = (q_1, \dots, q_n)$  be local coordinates on  $L$  and let  $p = (p_1, \dots, p_n)$  be dual coordinates on the fibers of  $T^*L$ . Write  $\partial_j = \frac{\partial}{\partial q_j}$  and  $\partial_{j^*} = \frac{\partial}{\partial p_j}$  for the corresponding tangent vectors. In order to shorten notation, we employ the summation convention in what follows. Let  $g$  be a metric on  $L$ ,  $g(q) = g_{ij}(q) dq_i \otimes dq_j$  and let  $g^{ij}(q)$  be such that  $g^{ij}(q) g_{jk}(q) = \delta_k^i$ , where  $\delta_k^i$  is the Kronecker delta. Let  $\Gamma_{jk}^i$  be the Christoffel symbols of  $g$  given by

$$\Gamma_{jk}^i(q) = \frac{1}{2} \sum_s g^{is} (\partial_k(g_{js}) + \partial_j(g_{ks}) - \partial_s(g_{jk})).$$

Then, the induced covariant derivative on the cotangent bundle is

$$\nabla(a_i dq_i) = (\partial_j(a_k) - \Gamma_{jk}^i a_i) dq_j \otimes dq_k.$$

This covariant derivative gives a decomposition of the tangent bundle  $T(T^*L)$  into a vertical sub-bundle  $V$  given by the kernel of the differential of the projection  $\pi: TL \rightarrow L$  and the horizontal sub-bundle  $H$  with fiber at  $\alpha \in T^*L$  spanned by tangent vectors at the starting

point of covariantly constant curves with initial value  $\alpha$ . In local coordinates

$$V_{(q,p)} = \text{Span} \{ \partial_{j^*} \}_{j=1}^n, \quad H_{(q,p)} = \text{Span} \{ \partial_j + p_r \Gamma_{js}^r \partial_{s^*} \}_{j=1}^n.$$

The almost complex structure  $J$  induced by  $g$  satisfies  $J(V) = H$  and is defined as follows on vertical vectors  $v \in V_\alpha$ : translate  $v$  to the origin of  $T_{\pi(\alpha)}^* L$ , identify the translate with a tangent vector of  $L$  using the metric, and let  $Jv$  be the negative of the horizontal lift of this tangent vector. In local coordinates:

$$(6.1) \quad J\partial_{j^*} = -g^{kj}(\partial_k + p_r \Gamma_{ks}^r \partial_{s^*}), \quad J\partial_j = g_{jk}\partial_{k^*} + g^{ls}(p_r \Gamma_{js}^r \partial_l + p_r p_v \Gamma_{js}^r \Gamma_{lt}^v \partial_{t^*}).$$

*Proof of Lemma 2.2.* Using Darboux balls around the double points of  $\Pi_P(L)$ , we may identify these neighborhoods with small balls around 0 in  $\mathbb{C}^n$ . Let  $J_0$  in these balls correspond to the standard almost complex structure. It is then easy to find a small Legendrian isotopy which makes both sheets at each intersection point agree with their tangent spaces at 0. Note that in a neighborhood of the intersection point, the pull-back of  $J_0$  under the neighborhood map which identifies cotangent fibers with Lagrangian subspaces perpendicular the image linear subspace correspond to the almost complex structure induced by a flat metric, see Remark 6.1. Letting  $g$  be a metric which is flat in these neighborhoods we may push forward the corresponding almost complex structure to a neighborhood of  $\Pi_P(L)$  using a symplectic neighborhood map which agrees with the map discussed above near each Reeb chord endpoint. This gives an almost complex structure in a neighborhood of  $\Pi_P(L)$  compatible with  $d\theta$ . Extending it to all of  $P$  establishes the first statement of the theorem.

The proof of the second statement is a word by word repetition of the proof of Proposition 2.3 in [8] once we establish transversality within the smaller class of almost complex structures used here (the additional condition here is that they are required to be standard in some neighborhood of  $\Pi_P(L)$ ) and an upper bound on dimension.

We start with the dimension bound. Since the area of any non-trivial  $J$ -holomorphic disk is bounded from below and since there are only finitely many Reeb chords, it follows that there are only finitely many Reeb chord collections  $(a; b_1, \dots, b_k)$  such that  $\mathcal{M}_A(a; b_1, \dots, b_k)$  may be non-empty. Gromov compactness<sup>3</sup> then implies that for each fixed Reeb chord collection, there are at most finitely many homology classes  $A$  and homotopy classes  $\alpha$  such that  $\mathcal{M}_A^\alpha(a; b_1, \dots, b_k)$  may be non-empty. This gives the uniform dimension bound.

As for transversality, in the argument in the proof of surjectivity in [8, Lemma 4.5 (1)], the perturbation  $K_S$  may be taken to have support disjoint from  $\Pi_P(L)$ . Hence, it follows from Gromov compactness of the moduli space of  $J$ -holomorphic disks and from the uniform area bound of the disks discussed above that we can achieve transversality among  $J$  which agree with  $J_0$  in some neighborhood  $N$  of  $\Pi_P(L)$ .  $\square$

*Proof of Lemma 2.5.* The proof follows from an argument entirely analogous to the proof of [5, Theorem 7.12]: consider the linearized  $\bar{\partial}_J$ -operator at a solution  $u: D \rightarrow P$  with boundary on  $L_0 \cup L_1(\hat{f})$ . The two positive punctures of  $u$  map to different points in  $\Pi_P(L_0) \cap \Pi_P(L_1(\hat{f}))$  since at a double point corresponding to a positive puncture the disk comes in along the lower branch and out along the upper. Thus, by adding perturbations of  $\hat{f}$  supported near the double point at one of the positive corners of  $u$ , we show that the operator is surjective on the complement of exceptional holomorphic disks. Exactly as in

<sup>3</sup>See [5, 8] for proofs of Gromov compactness in the present setting.

the proof of [5, Theorem 7.12], it follows that the existence of exceptional disks of small dimension contradicts this surjectivity result for disks of dimensions  $\leq 0$ . The lemma follows.  $\square$

**6.1.2. Broken disks.** By Gromov compactness, a sequence of elements in  $\mathcal{M}_A(a; b_1, \dots, b_k)$  has a subsequence which converges to a broken  $J$ -holomorphic disk with one positive puncture. As we shall see, the smooth moduli spaces corresponding to the pieces of the broken disks fit together to a compact manifold with boundary with corners  $\overline{\mathcal{M}}(a; b_1, \dots, b_k)$ , the interior of which is  $\mathcal{M}_A(a; b_1, \dots, b_k)$ .

A *broken  $J$ -holomorphic disk* with positive puncture at a Reeb chord  $a$  and negative punctures at the Reeb chords  $b_1, \dots, b_k$  is a collection of  $J$ -holomorphic disks  $u^0, \dots, u^r$  together with a connected rooted (and hence directed) planar tree  $T$  that has  $r + k + 2$  vertices, of which 1 is the root and another  $k$  are leaves. We will frequently drop  $T$  from the notation in the future. To each interior vertex of  $T$  there corresponds one disk  $u^j$ , with  $u^0$  adjacent to the root, and to each leaf there corresponds a Reeb chord  $b_j$  in counterclockwise order from the left of the root. These correspondences satisfy:

- Each disk  $u^j$  has one positive puncture and as many negative punctures as there are incoming edges into the vertex  $j$ .
- If there is an interior edge from vertex  $j$  to vertex  $l$ , and that edge is the  $m^{\text{th}}$  incoming edge counterclockwise from the outgoing edge of the vertex  $l$ , then the positive puncture of  $u^j$  and the  $m^{\text{th}}$  negative puncture of  $u^l$  both map to the same Reeb chord  $q$ . We say that  $u^j$  is *attached* to  $u^l$  at  $q$ .
- If there is a leaf labeled  $b_j$  whose outgoing edge goes to vertex  $l$ , and that edge is the  $m^{\text{th}}$  incoming edge counterclockwise from the outgoing edge of the vertex  $l$ , then the  $m^{\text{th}}$  negative puncture of  $u^l$  maps to the Reeb chord  $b_j$ .
- The positive puncture of  $u^0$  maps to the Reeb chord  $a$ .

See Figure 7. Let  $T^j$  be the subtree of  $T$  obtained by detaching the outgoing edge from  $u^j$  from its endpoint and adding a new root. We also remark that if  $u^0, \dots, u^r$  are the pieces of a broken  $J$ -holomorphic disk such that the boundary of  $u^j$  correspond to the homology class  $A^j \in H_1(L)$  then the boundary of the disk obtained by attaching all disks of the broken disk corresponds to  $A = \sum_j A^j$ .

**6.1.3. A gluing lemma.** Let  $u^0, \dots, u^r$  be a broken  $J$ -holomorphic disk with negative punctures at Reeb chords  $b_1, \dots, b_k$ , let  $u^j$  represent an element in  $\mathcal{M}^j = \mathcal{M}_{A^j}(a^j; b_1^j, \dots, b_{k_j}^j)$ , and write  $\mathcal{M} = \mathcal{M}_A(a; b_1, \dots, b_k)$ , where  $A = \sum_j A^j$ . In order to equip  $\overline{\mathcal{M}}_A(a, b_1, \dots, b_k)$  with the structure of a manifold with boundary with corners, we first construct a map

$$(6.2) \quad \Phi: \mathcal{M}^0 \times \dots \times \mathcal{M}^r \times [0, \infty)^r \rightarrow \mathcal{M}.$$

The map  $\Phi$  is defined by gluing the pieces of the broken disk into a nearby holomorphic disk. To construct the map, we will pass to a more general functional analytic setup so that we can apply the following result (known as Floer's Picard lemma):

**Lemma 6.2.** *Let  $f: B_1 \rightarrow B_2$  be a smooth Fredholm map of Banach spaces,*

$$(6.3) \quad f(v) = f(0) + df(v) + N(v).$$

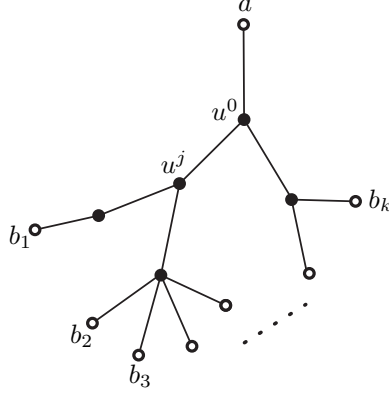


FIGURE 7. The tree structure of a broken disk with positive puncture at  $a$  and negative punctures at  $b_1, \dots, b_k$ .

Assume that  $df$  is surjective and has a bounded right inverse  $Q$  and that the non-linear term  $N$  satisfies a quadratic estimate of the form

$$(6.4) \quad \|N(u) - N(v)\|_{B_2} \leq C\|u - v\|_{B_1}(\|u\|_{B_1} + \|v\|_{B_1}).$$

If  $\|Qf(0)\| \leq \frac{1}{8C}$ , then for  $\epsilon < \frac{1}{4C}$ ,  $f^{-1}(0) \cap B(0; \epsilon)$ , where  $B(0; \epsilon)$  is an  $\epsilon$ -ball around  $0 \in B_1$ , is a smooth submanifold diffeomorphic to the  $\epsilon$ -ball in  $\text{Ker}(df)$ .

*Proof.* The proof appears in Floer [12]. Here we give a short sketch pointing out some features that we will use below. Let  $K = \text{Ker}(df)$  and choose a splitting  $B_1 = B'_1 \oplus K$  with projection  $\pi: B_1 \rightarrow K$ . For  $k \in K$ , define  $\hat{f}: B_1 \rightarrow B_2 \oplus K$ ,  $\hat{f}(b_1) = (f(b_1), \pi(b_1) - k)$ . Then solutions to the equation  $f(b_1) = 0$  with  $\pi(b_1) = k$  are in 1-1 correspondence with solutions to the equation  $\hat{f}(b_1) = 0$ . Moreover, the differential  $d\hat{f}$  is an isomorphism with inverse  $\hat{Q}$ .

On the other hand, solutions of the equation  $\hat{f}(v) = 0$  are in 1-1 correspondence with fixed points of the map  $F: B_1 \rightarrow B_1$  given by

$$F(v) = v - \hat{Q}\hat{f}(v).$$

To produce fixed points the Newton iteration scheme is applied: if

$$v_0 = k, \quad v_{j+1} = v_j - \hat{Q}\hat{f}(v_j),$$

then  $v_j$  converges to  $v_\infty$  as  $j \rightarrow \infty$  and  $F(v_\infty) = v_\infty$ . Furthermore, if  $\|f(0)\|$  is sufficiently small then there is  $0 < \delta < 1$  such that:

$$\|v_{j+1} - v_j\| \leq \delta^j \|f(0)\|$$

and consequently

$$(6.5) \quad \|v_\infty - v_0\| \leq M\|f(0)\|,$$

where  $M$  is a constant. □

6.1.4. *Functional analytic setup.* The functional analytic setup we use is that of a bundle of Banach manifolds of maps from the disk with boundary punctures into  $P$  with boundary on  $\Pi_P(L)$  and with punctures mapping to double points of  $\Pi_P(L)$ . The base of the bundle is the space of conformal structures on that disk; see [8, Section 3.1]. A neighborhood of a given map  $w$  in this bundle is parametrized by an exponential map acting on the product of a weighted Sobolev space  $\mathcal{H}_{2,\delta}$  of vector fields along  $w$  and a finite dimensional space of conformal variations  $V_{\text{con}}$  of the source. In this setting, the  $\bar{\partial}_J$ -operator gives a Fredholm section of the bundle of complex anti-linear maps from the tangent space of the source disk of  $w$  to the pull-back  $w^*(TP)$ . If  $\bar{\partial}_J(w) = 0$  then the zero-set of this section gives a neighborhood of  $w$  in the moduli space.

To better understand the setup, let us discuss the domains of the disks in greater detail. In order to gain better control of the map  $\Phi$  in (6.2), we add extra marked points to disks as follows. Let  $u: D_{m+1} \rightarrow P$  be a  $J$ -holomorphic disk with boundary on  $L$  and with positive puncture at the Reeb chord  $c$  with endpoints  $c^\pm \in L$ . Let  $S_{c^\pm}^{\text{in}} \subset L$  and  $S_{c^\pm}^{\text{out}} \subset L$  be small concentric spheres in  $L$  centered at  $c^\pm$  where  $S_{c^\pm}^{\text{out}}$  lies outside  $S_{c^\pm}^{\text{in}}$ . Since  $L$  is normalized at double points, elementary Fourier analysis can be used in a straightforward manner to derive asymptotics of  $u$  near double points of  $\Pi_P(L)$ . The asymptotics show that if  $S_{c^\pm}^{\text{in/out}}$  are chosen small enough then for an open dense set of radii there are points  $p_\pm^{\text{in/out}} \in \partial D_{m+1}$  which lie close to the positive puncture of  $u$  such that  $u(p_\pm^{\text{in/out}}) \in S_{c^\pm}^{\text{in/out}}$  and such that the intersections are transverse. Consequently, there are such points for each  $J$ -holomorphic  $v$  in some neighborhood of  $u$ . Consider  $p_\pm^{\text{in/out}}$  as punctures in the domain which are required to map to  $S_{c^\pm}^{\text{in/out}}$ , i.e. we view  $u$  as a map  $u: D_{m+5} \rightarrow P$ . In [7, Section 4.2.3] and [5, Section 8.6] it is explained how to describe a neighborhood of  $u$  in the moduli space in terms of disks with such extra marked points corresponding to intersections. We will use such disks with extra marked points below. Note that we add four marked points to each disk, which makes the domain of any disk stable.

Consider a broken disk with pieces  $u^0, \dots, u^k$  as above. We represent the domain of a disk  $u^j$  as the upper half plane  $H = \{x + iy \in \mathbb{C}: y \geq 0\}$  with marked points on the boundary in the following way: the puncture of  $H$  at  $\infty$  corresponds to the positive puncture of  $u^j$  and we take the punctures closest to the positive puncture (i.e. the punctures corresponding to  $p_\pm^{\text{in}}$ ) to lie at  $x = \pm 1$ . Then the location of remaining punctures in the interval  $(-1, 1) \subset \mathbb{R} = \partial H$  determines the conformal structure on the domain of  $u^j$  uniquely. We will equip such punctured half planes with a special metric for which the neighborhood of any puncture looks like a half infinite strip. More precisely, at the positive puncture we use the map  $(-\infty, 0] \times [0, 1] \rightarrow H$ ,  $\tau + it \mapsto 2e^{-\pi(\tau+it)}$  to identify the neighborhood with a half strip, at other punctures  $q$  we fix  $r_0 > 0$  sufficiently small so that the interval on the real axis of length  $2r_0$  centered at  $q$  does not contain other punctures and use the map  $[0, \infty) \times [0, 1] \rightarrow H$ ,  $\tau + it \mapsto q + r_0 e^{-\pi(\tau+it)}$  to identify the neighborhood with a half strip. We use a Riemannian metric on this domain which agrees with the standard metric on the half strips  $(-\infty, -1] \times [0, 1]$  (respectively  $[1, \infty) \times [0, 1]$ ) and which interpolates smoothly to the flat metric on the compact part of  $H$  which is the complement of the open half strip neighborhoods. We denote the domain of  $u^j$  by  $\Delta^j$ .

We now give a more precise definition of the space of conformal variations  $V_{\text{con}}^j$  of the source of  $u^j$ : conformal variations of the domain of  $u^j$  are linearizations of diffeomorphisms which move the punctures along the boundary. We take  $V_{\text{con}}^j$  to be the space spanned by conformal variations supported near all punctures except the positive puncture and the two punctures closest to it (these two correspond to  $p_{\pm}^{\text{in}}$ ). In terms of half strip coordinates, a conformal variation of the source has the form  $\bar{\partial}(\beta v)$  where  $\beta$  is a smooth cut-off function equal to 1 near the puncture and equal to 0 on remaining parts of the disk, and where  $v$  is the holomorphic vector field given by  $e^{\pi z}$  in the half strip  $[0, \infty) \times [0, 1]$ .

6.1.5. *Approximately holomorphic maps.* In order to define  $\Phi(u^0, \dots, u^r, \rho)$ , where  $u^j \in \mathcal{M}^j$  and where  $\rho = (\rho_1, \dots, \rho_r) \in [0, \infty)^r$  has all components  $\rho_j$  sufficiently large, we first define a domain  $\Delta(\rho)$  which is conformally equivalent to the upper half plane with punctures on the boundary but which has a metric somewhat different from the metrics on  $\Delta^j$ . Once we have that domain, we will construct an approximately holomorphic map on  $\Delta(\rho)$ .

The punctures on a domain  $\Delta^j$  are of two types: punctures corresponding to interior edges of the tree  $T$  and those corresponding to edges coming from leaves. We call the first type of punctures *bound punctures* and the second *free punctures*. Let  $\hat{\Delta}^0(\rho)$  denote the subset of  $\Delta^0$  obtained as follows: for each bound puncture  $q$  in  $\Delta^0$ , cut off  $(\rho_j, \infty) \times [0, 1]$  from its half strip neighborhood, where the index  $j$  corresponds to the index of the disk  $u^j$  attached at  $q$ . For  $j > 0$ , let  $\hat{\Delta}^j(\rho)$  denote the subset of  $\Delta^j$  obtained as follows: at the positive puncture cut off  $(-\infty, -\rho_j + 1) \times [0, 1]$  and for each bound puncture  $q$  in  $\Delta^j$ , cut off  $(\rho_k, \infty) \times [0, 1]$  from its half strip neighborhood, where the index  $k$  corresponds to the index of the disk  $u^k$  attached at  $q$ .

Join the domains  $\hat{\Delta}^j(\rho)$  to the domain  $\Delta(\rho)$  by identifying the vertical part of boundary of  $\hat{\Delta}^j(\rho_j)$ ,  $j \geq 1$ , which lies in the half strip corresponding to the positive puncture with the vertical part of the boundary of  $\hat{\Delta}^l(\rho)$  lying in the half strip of the negative puncture at which the positive puncture of  $\Delta^j$  is attached. Since metrics agree in gluing regions, this gives a domain with a metric. In order to determine the conformal structure we note that if we view all domains as upper half planes with punctures on the boundary then the above construction corresponds to cutting out half-disks of radii  $e^{-\pi\rho_j}$  near bound punctures and inserting scaled cut off half planes. This construction is repeated so that further scaled half planes are attached in scaled half planes already attached; see Figure 8.

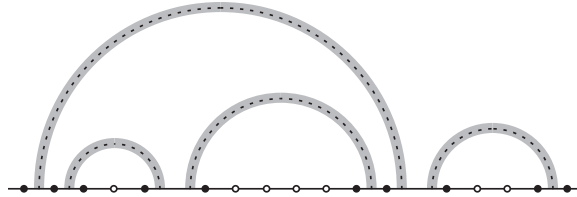


FIGURE 8. The glued domain for the broken disk in Figure 7. Shaded regions correspond to the half strips near the punctures. Stationary punctures are filled, moving punctures are not.

Consider next the space of conformal variations  $V_{\text{con}}(\rho)$  of  $\Delta(\rho)$ . We will use a decomposition of this space of the form

$$V_{\text{con}}(\rho) = V'_{\text{con}}(\rho) \oplus V''_{\text{con}}(\rho).$$

To define the decomposition we first note that  $V_{\text{con}}(\rho)$  is spanned by conformal variations as described above supported near each puncture except at the positive puncture and at the two punctures closest to it. These punctures correspond to the positive puncture of  $u^0$  and the intersection points  $p_{\pm}^{\text{in}}$  of  $u^0$ . Other punctures of the disk  $\Delta(\rho)$  are of two types:

- *Moving punctures* which correspond to (free) punctures in  $\Delta^j$  where some conformal variation in  $V_{\text{con}}^j$  was supported; see Figure 8.
- *Stationary punctures* which correspond to (free) punctures in  $\Delta^j$  where no conformal variation in  $V_{\text{con}}^j$  was supported; see Figure 8.

Second, we use the tree structure on the broken disk. Let  $T$  be the directed tree corresponding to the broken disk. We take  $V'_{\text{con}}(\rho)$  generated by the conformal variations at all moving punctures and the following additional variations  $\gamma^j$ . Consider the conformal representative of  $\Delta(\rho)$ , which is an upper half plane  $H(\rho)$  with boundary punctures. Let  $u^j$  be a disk. For large  $\rho_j$  the punctures of  $\Delta(\rho)$  corresponding to the negative punctures of  $u^k$  for all  $u^k \in T^j$  lie very close together and we let  $\gamma^j$  denote the conformal variation which correspond to moving all these punctures uniformly along the boundary  $\partial H$ . In terms of the generators of  $V_{\text{con}}(\rho)$  this is a linear combination of the variations at all stationary and moving punctures coming from  $\Delta^k$  for  $u^k \in T^j$ . Finally, we take  $V''_{\text{con}}(\rho)$  to be generated by the conformal variations  $\theta_j$  which correspond to moving all punctures corresponding to negative punctures of  $u^k \in T^j$  toward the mid-point of the subinterval  $I^j \subset \mathbb{R} = \partial H$  bounded by the punctures corresponding to  $p_{\pm}^{\text{in}}$  in  $\Delta^j$ , by scaling and which keeps all other punctures fixed.

Consider  $H(\rho)$  with boundary punctures as just described and define

$$d(\rho, j) = d_+^{\text{in}}(\rho, j) + d_-^{\text{in}}(\rho, j),$$

where  $d_{\pm}^{\text{in}}(\rho, j)$  is the distance between the puncture corresponding to  $p_{\pm}^{\text{in}}(\rho, j)$  and the puncture at  $\pm 1 \in \mathbb{R} = \partial H$ . Then all conformal variations in  $V'_{\text{con}}(\rho)$  preserve  $d(\rho, j)$  for  $j = 1, \dots, r$ . Furthermore, note that if  $\tau = (0, \dots, x, \dots, 0)$  where  $x > 0$  sits in the  $j^{\text{th}}$  component and if  $\rho' = \rho + x$  then  $d(\rho', k) > d(\rho, k)$  for all  $k$  such that  $u^k \in T^j$ .

Define the approximately holomorphic map

$$w_{u^0, \dots, u^r; \rho}: \Delta(\rho) \rightarrow P,$$

by interpolating between the joined maps in the region  $[\rho + 1, \rho] \times [0, 1]$  as in [8, Proof of Proposition 4.6].

**6.1.6. Definition and properties of the corner map.** We apply Lemma 6.2 to produce solutions near  $w_{u^0, \dots, u^r; \rho}$  for  $\rho$  sufficiently large. We take  $f$  to be a restriction of the  $\bar{\partial}_J$ -operator in local coordinates centered at  $w_{u^0, \dots, u^r; \rho}$ . More precisely, trivializing the bundle of complex anti-linear maps in a neighborhood of  $w_{u^0, \dots, u^r; \rho}$  as described in [8, Section 3.2] and using a trivialization of the tangent bundle of the disk, we view the  $\bar{\partial}_J$ -operator as a map

$$\bar{\partial}_J: \mathcal{H}_{2, \delta} \oplus V_{\text{con}}(\rho) \rightarrow \mathcal{H}_{1, \delta},$$

where  $\mathcal{H}_{1,\delta}$  is the Sobolev space of vector fields along  $w_{u^0,\dots,u^r;\rho}$  with one derivative in  $L^2$  weighted by the same weight function used to define  $\mathcal{H}_{2,\delta}$ . As in [8, Proof of Proposition 4.6], we find:

$$(6.6) \quad \|\bar{\partial}_J w_{u^0,\dots,u^r;\rho}\|_{1,\delta} = \mathbf{O}(e^{-\theta_0 \min_j \rho_j}),$$

where  $\|\cdot\|_{1,\delta}$  is the norm on  $\mathcal{H}_{1,\delta}$ , for some  $\theta_0 > 0$  determined by the complex angles (see [8, Section 4.1]) at the double points of  $\Pi_P(L)$ .

We take  $f$  of Lemma 6.2 as the restriction of  $\bar{\partial}_J$  to the subspace  $\mathcal{H}_{2,\delta} \oplus V'_{\text{con}}(\rho)$  so that

$$f = \bar{\partial}_J: \mathcal{H}_{2,\delta} \oplus V'_{\text{con}}(\rho) \rightarrow \mathcal{H}_{1,\delta},$$

and by (6.6),  $\|f(0)\|_{1,\delta} = \mathbf{O}(e^{-\theta_0 \min_j \rho_j})$

**Lemma 6.3.** *There exists  $\rho_0 > 0$  such that for  $\rho = (\rho_1, \dots, \rho_r) \in [\rho_0, \infty)^r$ , the differential  $df$  is surjective and admits a bounded right inverse. Moreover, the quadratic estimate (6.4) for the non-linear term in the Taylor expansion of  $f$  holds.*

*Proof.* The quadratic estimate follows from [8, Proof of Proposition 4.6]. In order to see that the differential is surjective and admits a bounded right inverse, we first note that it is a Fredholm operator of index  $d = \sum_j \dim(u^j)$ . Second, consider the subspace  $K$  of cut-off kernel functions corresponding to all pieces of the disk. It follows that  $\dim(K) = d$ . Note that the component of the cut off kernel element in the disk  $\Delta^j$  along the conformal variation supported at a negative puncture where the disk  $\Delta^l$  is attached corresponds to the conformal variation  $\gamma^j$ . Now a standard argument (see, for example, [5, Lemma 8.9]) shows that on the subspace which is the  $L^2$ -complement of  $K$ , there is an estimate

$$\|df(v)\| \geq C\|v\|,$$

provided the components of  $\rho$  are sufficiently large.  $\square$

With Lemma 6.3 established, define  $\Phi_\rho(u^0, \dots, u^r; \rho)$  as the result of applying Newton iteration to  $w_{u^0,\dots,u^r;\rho}$  as in the proof of Lemma 6.2. Combining the above maps for all  $\rho \in [\rho_0, \infty)^r$  with sufficiently large  $\rho_0$  and reparametrizing  $[0, \infty) \approx [\rho_0, \infty)$ , we get the map

$$\Phi: \mathcal{M}^0 \times \dots \times \mathcal{M}^r \times [0, \infty)^r \rightarrow \mathcal{M}.$$

**Lemma 6.4.** *The Newton iteration map  $\Phi$  is an embedding.*

*Proof.* Let  $u = (u^0, \dots, u^k)$ . We first look at the conformal structure of the domain of  $\Phi(u; \rho)$ : by construction  $d^j(\rho)$  is independent of  $u$ . If  $\rho \neq \rho'$  then let  $j$  be an index such that the  $\rho_j \neq \rho'_j$  and such that  $\rho_l = \rho'_l$  for all  $l \neq j$  with  $u^j \in T^l$  and it follows that the conformal structures of  $\Phi(u; \rho)$  and  $\Phi(u'; \rho')$  are different if  $\rho \neq \rho'$ .

It is then a consequence of Lemma 6.2 that  $\Phi$  is a local embedding and, if  $\Phi(u; \rho) = \Phi(u'; \rho)$ , then the estimate (6.5) implies that  $w_{u';\rho}$  lies in a small neighborhood of  $w_{u;\rho}$ . Thus,  $\Phi$  is an actual embedding.  $\square$

**6.1.7. Corner structure.** With Lemma 6.4 established, we can define the compactification of the moduli space  $\mathcal{M}$  by adding broken configurations at corners with the Newton iteration map as a local chart. To discuss this, assume that  $\mathcal{M}$  and  $\mathcal{M}^0, \dots, \mathcal{M}^r$  are as in Lemma 6.4 and let

$$\hat{\Phi}: \mathcal{M}^0 \times \dots \times \mathcal{M}^r \times [0, \infty)^r \rightarrow \mathcal{M},$$

denote the Newton iteration embedding. Assume further that for each  $j = 1, \dots, r$  there are broken disks in  $\mathcal{M}^{j;1} \times \dots \times \mathcal{M}^{j;k_j}$  which can be joined to a disk in  $\mathcal{M}^j$ . Let

$$\Phi^j: \mathcal{M}^{j;0} \times \dots \times \mathcal{M}^{j;k_j} \times [0, \infty)^{k_j} \rightarrow \mathcal{M}^j, \quad j = 0, \dots, r,$$

denote the corresponding Newton iteration maps. Finally, Newton iteration can be applied directly to broken disks in

$$(\mathcal{M}^{0;0} \times \dots \times \mathcal{M}^{0;k_0}) \times \dots \times (\mathcal{M}^{r;0} \times \dots \mathcal{M}^{r;k_r})$$

and produces disks in  $\mathcal{M}$ . Let

$$\begin{aligned} \Phi: \mathcal{M}^{0;0} \times \dots \times \mathcal{M}^{0;k_0} \times \dots \times \mathcal{M}^{r;0} \times \dots \times \mathcal{M}^{r;k_r} \\ \times [0, \infty)^{k_0} \times \dots \times [0, \infty)^{k_r} \times [0, \infty)^r \rightarrow \mathcal{M}, \end{aligned}$$

denote the corresponding embedding.

Combining the maps  $\Phi^j$  and  $\widehat{\Phi}$ , we get an embedding:

$$\Psi: \mathcal{M}^{0;0} \times \dots \times \mathcal{M}^{0;k_0} \times \dots \times \mathcal{M}^{r;0} \times \dots \times \mathcal{M}^{r;k_r} \times [0, \infty)^{k_0} \times \dots \times [0, \infty)^{k_r} \times [0, \infty)^r \rightarrow \mathcal{M},$$

with  $\tilde{\rho} = (\rho_0, \dots, \rho_r) \in [0, \infty)^{k_0} \times \dots \times [0, \infty)^{k_r}$  and  $\rho \in [0, \infty)^r$

$$\begin{aligned} \Psi(u^{0;0}, \dots, u^{0;k_0}, \dots, u^{r;0}, \dots, u^{r;k_r}; \tilde{\rho}, \rho) = \\ \widehat{\Phi}(\Phi^0(u^{0;0}, \dots, u^{0;k_0}; \rho^0), \dots, \Phi^r(u^{r;0}, \dots, u^{r;k_r}; \rho^r); \rho). \end{aligned}$$

We claim that the maps  $\Psi \circ \Phi^{-1}$  and  $\Phi \circ \Psi^{-1}$  are  $C^1$  diffeomorphisms. To see this consider the definitions of the gluing maps involved and note that  $L^2$ -projections define linear isomorphisms between the spaces spanned by cut off kernel functions of the differentials on pieces of a broken disk and the kernel of the differential of the approximately holomorphic disk which is the starting point for the Newton iteration of Lemma 6.2; see the proof of Lemma 6.3. Since the kernel of the differential at the approximately holomorphic disk is the domain of the chart, the composition statements follow since the corresponding statements on the level of cut off kernel functions clearly hold.

In order to define the structure of a manifold with boundary with corners identify  $[0, \infty)$  with  $[0, 1)$ . Consider the inclusion  $[0, 1) \subset [0, 1]$ , and complete the map  $\Phi$  to a chart

$$\overline{\Phi}: \mathcal{M}^0 \times \dots \times \mathcal{M}^r \times [0, 1]^r \rightarrow \mathcal{M}$$

by identifying the points  $(u^0, \dots, u^r; \rho)$  with  $\rho = (\rho_1, \dots, \rho_r)$ , where some of the  $\rho_j$  equal 1, as follows. Consider the tree  $T$  corresponding to the broken disk  $u^0, \dots, u^r$ . For each  $s$  such that  $\rho_s = 1$ , replace each edge from the vertex  $s$  to the vertex  $j$  by two edges, one of which connects  $s$  to a new root and the other of which connects  $j$  to a new leaf that has the same Reeb chord label as the new leaf. For edges leaving  $s$ , the role of roots and leaves are reversed. See Figure 9. Each component of the new graph corresponds to a broken disk  $u^{l;0}, \dots, u^{l;l_k}$  and  $\rho^l = (\rho_{l;0}, \dots, \rho_{l;l_s})$ , where  $\rho^{l;j}$  is the component of  $\rho$  corresponding to  $u^{l;j}$ . There is a corresponding chart

$$\Phi^l: \mathcal{M}^{l;0} \times \mathcal{M}^{l;l_s} \times [0, 1]^{l_s} \rightarrow \mathcal{M}^l.$$

Identify the point  $(u^0, \dots, u^r; \rho)$  with the broken disk with components  $\Phi^l(u^{l;0}, \dots, u^{l;l_s}; \rho^l)$ ,  $l = 1, \dots, k$  and use the coordinate chart  $\Phi$  which glues this configuration to define a

coordinate neighborhood. The compatibility between  $\Phi$  and  $\Psi$  discussed above then ensures that this construction gives a well-defined  $C^1$ -structure.

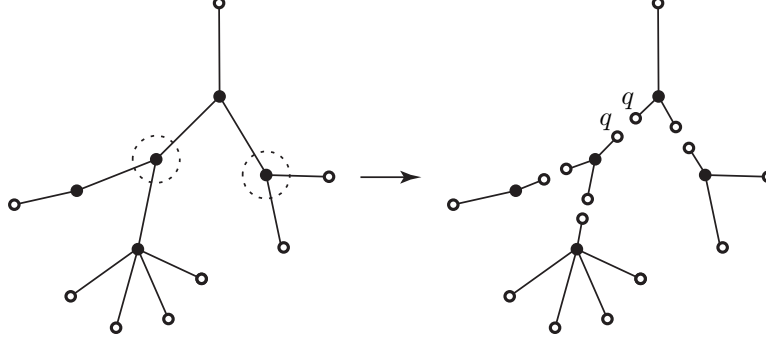


FIGURE 9. Splitting the broken disk into sub-disks when  $\rho_s = 1$ .

In order to build the compactified moduli space, we use the area filtration. The moduli space of disks of smallest area is a compact manifold since no disk in this space can break into smaller pieces. In fact, the moduli space of disks of second smallest area is a compact manifold as well since a disk with only one positive puncture cannot be glued to itself. In order to define the boundaries of moduli spaces of disks of a given area  $A$  we use the gluing maps discussed above applied to moduli spaces of disks of area smaller than  $A$ . The boundaries of the smaller area moduli spaces were constructed in previous steps and by the compatibility discussed above this construction yields a manifold with boundary with corners of class  $C^1$ . Note though that the  $C^1$ -structure is non-canonical since it depends on our specific choices of conformal models of the domains, on the interpolations between maps, and on the choice of identification of  $[0, \infty)$  with  $[0, 1)$ .

**6.2. Evaluation maps.** The generalized disks that appear in Theorem 3.6 require the use of evaluation maps on compactified moduli spaces of  $J$ -holomorphic disks which in turn requires a slight extension of the definition of compactified moduli space above. As in Subsection 6.1, we add marked points near positive punctures in order to get stable domains of all  $J$ -holomorphic disks. We then add one additional marked point on the boundary. If  $\mathcal{M}$  is the original moduli space, we let  $\mathcal{M}^*$  denote the corresponding moduli space of disks with one extra marked point on the boundary. Note that we have a fibration  $\mathcal{M}^* \rightarrow \mathcal{M}$ . Furthermore, there is a natural evaluation map

$$\text{ev}: \mathcal{M}^* \rightarrow L,$$

which corresponds to evaluation of the boundary lift of  $u$  at the marked point and this map has a natural extension to configurations where the extra marked point agrees with one of the extra punctures by evaluating at the puncture. It is straightforward to extend the notion of a disk with marked point on the boundary to broken configurations: here the marked point is in one of pieces of the disk. Furthermore, the gluing map  $\Phi$  discussed in Subsection 6.1 can be applied also to broken disks with marked points to give a disk with marked point as long as the marked point lies outside the piece which is cut off from the domain. However, for a fixed broken configuration with a marked point on the boundary,

there is a  $\rho_0$  such that if the components of  $\rho$  in Lemma 6.4 are all bigger than  $\rho_0$  then the map  $\Phi$  gives an embedding into the moduli space of disks with one extra marked point. Thus, there is a compactification  $\overline{\mathcal{M}}^*$  of the space  $\mathcal{M}^*$ , where the boundary has two types of strata: one corresponds to broken disks with one marked point, and the other to the marked point lying at Reeb chord endpoints. As in Subsection 6.1, this compactified space has the structure of a  $C^1$  manifold with boundary with corners and the evaluation maps fit together to give a  $C^1$ -map

$$\text{ev}: \overline{\mathcal{M}}^* \rightarrow \Pi_P(L).$$

See [15] for a discussion of evaluation maps in a similar situation.

### 6.3. Morse flows and generalized disks.

6.3.1. *Transversality.* The next step in proving Theorem 3.6 is to find triples  $(f, g, J)$  where  $f: L \rightarrow \mathbb{R}$  is a Morse function,  $g$  is a Riemannian metric on  $L$ , and  $J$  is an almost complex structure on  $P$  which have the following properties:

- (a1) The almost complex structure  $J$  is adjusted to  $L$  and  $L$  is normalized at double points; see Subsection 2.2.3.
- (a2) Condition (a1) implies that there are Reeb chord coordinates at the Reeb chord endpoints in  $L$  in which the Lagrangian projection is a linear embedding as a map into double point coordinates; see Subsection 2.2.3 for notation. The metric  $g$  agrees with the standard flat  $\mathbb{R}^n$  metric in Reeb chord coordinates.
- (a3) There is a symplectic neighborhood map  $\Phi: U \rightarrow P$ , where  $U \subset T^*L$  is a neighborhood of the 0-section, which is linear in the canonical coordinates corresponding to Reeb chord coordinates as a map into double point coordinates, and which is such that the almost complex structure on  $U$  induced by  $g$  is equal to  $\Phi^*J$ . (In other words,  $J$  is standard in a neighborhood of  $\Pi_P(L)$  with respect to  $g$ .)
- (a4) The Morse function  $f: L \rightarrow \mathbb{R}$  is real analytic and regular in the Reeb chord coordinate neighborhoods; see Lemma 2.5.
- (a5) If  $p$  is a critical point of  $f$ , then the eigenvalues of the Hessian of  $f$  at  $p$  (i.e. the critical values of the Hessian quadratic form restricted to the unit sphere in  $T_p L$  determined by  $g$ ) all have the same absolute value. If a Morse function has this property we say that it is *round at critical points with respect to  $g$* .

We call triples  $(f, g, J)$  with properties (a1) – (a5) *adjusted to  $L$* .

Let  $L \subset P \times \mathbb{R}$  be chord generic, let  $J$  be an almost complex structure, and let  $g$  be a Riemannian metric on  $L$  such that (a1) – (a3) above hold and such that Lemma 2.2 holds. Let  $\overline{\mathcal{M}}^*$  denote the moduli space of  $J$ -holomorphic disks with one positive puncture and with a marked point on the boundary and let  $\text{ev}: \overline{\mathcal{M}}^* \rightarrow L$  denote the evaluation map. If  $f: L \rightarrow \mathbb{R}$  is a Morse function such that the  $g$ -gradient of  $f$  defines a Morse-Smale flow and if  $p \in L$  is a critical point of  $f$ , then the stable and unstable manifolds  $W^s(p)$  and  $W^u(p)$  are submanifolds of  $L$  with natural compactifications consisting of broken (and constant) flow lines.

**Lemma 6.5.** *There exists a Morse function  $f: L \rightarrow \mathbb{R}$  with  $g$ -gradient which is Morse-Smale such that the triple  $(f, g, J)$  is adjusted to  $L$  and such that, for each critical point  $p$  of  $f$ ,  $W^s(p)$  and  $W^u(p)$  are stratumwise transverse to  $\text{ev}: \overline{\mathcal{M}}^* \rightarrow L$ . In particular, generalized disks defined by  $(f, g, J)$  have the following properties:*

- (g1) *There are no generalized disks of formal dimension  $< 0$ .*
- (g2) *All generalized disks of dimension 0 are transversely cut out.*

Furthermore,  $(f, g, J)$  can be chosen so that the following condition holds.

- (g3) *Rigid generalized disks defined by  $(f, g, J)$  intersect in general position in the following sense.*
  - *If  $\dim(L) > 2$  then the interior of the  $J$ -holomorphic disk part of any rigid generalized disk is disjoint from  $\Pi_P(L)$ , boundary lifts of two distinct  $J$ -holomorphic disks which are parts of rigid generalized disks are disjoint and if the boundary lifts of two rigid generalized disks intersect then these two generalized disks have either have the same disk part of flow line parts which are subsets of the same flow line.*
  - *If  $\dim(L) = 2$  then the interior of the  $J$ -holomorphic disk part of any rigid generalized disk is disjoint from the  $\Pi_P$ -image of the boundary lift of any rigid generalized disk, boundary lifts of two distinct  $J$ -holomorphic disks which are parts of two rigid generalized disks intersect transversely in finitely many points and if the boundary lifts of two rigid generalized disks intersect non-transversely or in infinitely many points, then these two generalized disks have either have the same disk part of flow line parts which are subsets of the same flow line.*

*Proof.* Fix a Morse function with Morse-Smale  $g$ -gradient for which (a4) and (a5) hold, and for which the critical points of  $f$  are in general position with respect to  $\text{ev}(\overline{\mathcal{M}}^*)$ . As in [29], linearized perturbations of  $f$  supported in small neighborhoods of spheres around the critical points span the normal bundles of all stable and unstable manifolds. Consequently, we may achieve the desired transversality by perturbation of  $f$  near the critical points. The technical details of this argument are analogous to [29], so we leave them out.  $\square$

We say that the *special points* of a rigid generalized disk consist of the junction points and critical points; in case  $\dim(L) = 2$ , we also add in the intersection points of lifts of rigid generalized disks. We call triples  $(f, g, J)$  which are adjusted to  $L$  and for which (g1) – (g3) hold *generic with respect to rigid generalized disks*.

**6.3.2. Normal Forms.** We next show that we can deform a triple  $(\hat{f}, \hat{g}, \hat{J})$  which is generic with respect to rigid generalized disks to a triple  $(f, g, J)$  which maintains those properties and has a certain normal form near special points. This will allow us to apply the techniques of [3] during our analysis of the correspondence between rigid generalized disks and  $J$ -holomorphic disks with boundary on the perturbed  $2L$ .

First, we add more special points: consider a flow line segment  $\gamma$  which connects a critical point and a junction point. Note that any such segment has finite length and add a finite number of points along  $\gamma$  so that they are of distance at most  $\eta > 0$  apart. Call these new points *special points* as well.

We say that a triple  $(f, g, J)$  which is generic with respect to rigid generalized disks is *semi-normalized* if the following conditions hold.

- (n1) If  $p$  is a critical point of  $f$  then there are coordinates  $x = (x_1, \dots, x_n)$  on a neighborhood  $U(p)$  around  $p$  in which  $f$  and  $g$  are given by

$$f(x) = c + \sigma_1 x_1^2 + \dots + \sigma_n x_n^2, \quad g(x) = \sum_j dx_j \otimes dx_j,$$

where  $|\sigma_j| = \sigma > 0$ ,  $j = 1, \dots, n$ .

- (n2) If  $q$  is a special point which is not a critical point (or an intersection point if  $\dim(L) = 2$ ) of a generalized disk, then there are coordinates  $x = (x_1, \dots, x_n)$  on a neighborhood  $U(q)$  around  $q$  in which  $f$  and  $g$  are given by

$$f(x) = c + \mu x_1, \quad g(x) = \sum_j dx_j \otimes dx_j,$$

for constants  $c, \mu$ , where  $\mu \neq 0$ .

Note that (g1) – (g3) of Lemma 6.5 hold for semi-normalized triples.

**Lemma 6.6.** *Let  $\hat{f}: L \rightarrow \mathbb{R}$  be a Morse function, let  $\hat{g}$  be a metric, and let  $\hat{J}$  be an almost complex structure such that the triple  $(\hat{f}, \hat{g}, \hat{J})$  is generic with respect to rigid generalized disks. Then there exist a Morse function  $f: L \rightarrow \mathbb{R}$ , a metric  $g$ , and an almost complex structure  $J$  such that the triple  $(f, g, J)$  is semi-normalized and has the following property. There is a  $\delta > 0$  and a bijective correspondence between the rigid generalized disks determined by  $(\hat{f}, \hat{g}, \hat{J})$  and rigid generalized disks determined by  $(f, g, J)$  such that there is exactly one rigid generalized disk determined by one of the triples with a lift in a  $\delta$ -neighborhood of a given rigid generalized disk determined by the other triple.*

*Proof.* We start by making a general remark about almost complex structures induced by Riemannian metrics. Let  $\hat{g}$  and  $g$  be Riemannian metrics on  $L$  and let  $\hat{J}$  respectively  $J$  denote the corresponding almost complex structures induced on  $T^*L$ . In order to discuss distances, we fix a reference metric  $h$  on  $L$  and note that it induces a metric on all tensor bundles of  $L$  as well as on  $T^*L$  and on its tensor bundles. Let  $d_{C^j}(T_1, T_2)$  denote the  $C^j$ -distance between tensor fields  $T_1$  and  $T_2$  as measured with respect to  $h$ . Consider a small  $\epsilon > 0$ . It follows from the local coordinate formula (6.1) that if  $d_{C^1}(\hat{g}, g) = \mathbf{O}(\epsilon)$  and if  $d_{C^2}(\hat{g}, g) = \mathbf{O}(1)$  then, in an  $\epsilon$ -neighborhood of the 0-section,

$$(6.7) \quad d_{C^0}(\hat{J}, J) = \mathbf{O}(\epsilon^2)$$

$$(6.8) \quad d_{C^1}(\hat{J}, J) = \mathbf{O}(\epsilon).$$

Consider a critical point  $p$  of  $\hat{f}$ . Choose normal coordinates in an  $\epsilon$ -ball around  $p$ . Define  $f$  to equal the second degree Taylor polynomial of  $\hat{f}$  in these coordinates in a neighborhood of 0 and define  $g$  by letting the metric be constant in a small neighborhood of 0. Then  $d_{C^2}(f, \hat{f}) = \mathbf{O}(\epsilon^3)$ ,  $d_{C^1}(\hat{g}, g) = \mathbf{O}(\epsilon)$ , and  $d_{C^2}(\hat{g}, g) = \mathbf{O}(1)$ . We conclude that stable and unstable manifolds determined by  $f$  and  $g$  are at  $C^1$ -distance  $\mathbf{O}(\epsilon)$  from those determined by  $\hat{f}$  and  $\hat{g}$ .

Let  $J$  be an almost complex structure which is standard with respect to  $g$  in an  $\epsilon$ -neighborhood of the 0-section and which is such that  $d_{C^1}(\hat{J}, J) = \mathbf{O}(\epsilon)$ ; see (6.8). If  $J'$  is an almost complex structure, write  $\mathcal{M}_{J'}$  for the moduli space of  $J'$ -holomorphic disks.

As in Subsection 6.1, we consider  $\mathcal{M}_{J'}$  as the 0-sets of a  $\bar{\partial}_{J'}$ -operator acting on a bundle of Sobolev spaces. In order to compare the evaluation maps  $\text{ev}: \overline{\mathcal{M}}_J^* \rightarrow L$  and  $\text{ev}: \overline{\mathcal{M}}_{J'}^* \rightarrow L$ , we use weighted Sobolev spaces  $W^{2,p}$  with two derivatives in  $L^p$ ,  $p > 2$ . Since  $d_{C^1}(\hat{J}, J) = \mathbf{O}(\epsilon)$ , we find that for  $\epsilon > 0$  small enough, the zero sets of  $\bar{\partial}_J$  and  $\bar{\partial}_{\hat{J}}$  are  $C^1$ -diffeomorphic and zeros of the  $\bar{\partial}_J$ - and  $\bar{\partial}_{\hat{J}}$ -operator are at distance  $\mathbf{O}(\epsilon)$  in the corresponding Sobolev norm. Since the  $W^{2,p}$ -norm controls the  $C^1$ -norm it follows, using this argument at each step in the inductive construction of the compactification of moduli spaces as manifolds with boundary with corners that the moduli spaces are  $C^1$ -diffeomorphic and that  $\text{ev}(\overline{\mathcal{M}}_J^*)$  and  $\text{ev}(\overline{\mathcal{M}}_{J'}^*)$  are  $C^1$ -close. Combining the fact that both Morse flows and evaluation maps determined by the triples  $(\hat{f}, \hat{g}, \hat{J})$  and  $(f, g, J)$ , respectively, are arbitrarily  $C^1$ -close with the genericity of  $(\hat{f}, \hat{g}, \hat{J})$  with respect to rigid generalized disks, we draw the following two conclusions. First, the triple  $(f, g, J)$  is generic with respect to rigid generalized disks. Second, there is a 1 – 1 correspondence between rigid generalized disks determined by  $(\hat{f}, \hat{g}, \hat{J})$  and those determined by  $(f, g, J)$  with properties as claimed in the statement.

Relabel the triple  $(f, g, J)$  just constructed and which has desired properties near critical points  $(\hat{f}, \hat{g}, \hat{J})$  and consider a special point  $q$  which is not a critical point. Choose normal coordinates in an  $\epsilon$ -ball around  $q$ , replace  $\hat{f}$  by its first degree Taylor polynomial in this ball, and let  $g$  be the flat metric in the normal coordinates. Since the flow time that a flow line spends in the region where the function is deformed is  $\mathbf{O}(\epsilon)$ , since the deformation of the vector field is  $\mathbf{O}(\epsilon)$  in  $C^0$ -norm and  $\mathbf{O}(1)$  in  $C^1$ -norm, and since the deformation of the metric is  $\mathbf{O}(\epsilon)$  in  $C^1$ -norm, we conclude that the resulting deformation of the Morse flow is  $\mathbf{O}(\epsilon^2)$  in  $C^0$ -norm and  $\mathbf{O}(\epsilon)$  in  $C^1$ -norm.

We next derive a corresponding result for the induced deformation of evaluation maps. First consider the moduli spaces as 0-sets in bundles of weighted Sobolev spaces  $W^{1,p}$  with one derivatives in  $L^p$ ,  $p > 2$ . Using (6.7), we find that the zero sets of  $\bar{\partial}_J$  and  $\bar{\partial}_{\hat{J}}$  are at distance  $\mathbf{O}(\epsilon^2)$  in the corresponding Sobolev norm. Since the  $W^{1,p}$ -norm controls the  $C^0$ -norm, the image of a smooth component of an evaluation map lies in an  $\mathbf{O}(\epsilon^2)$ -neighborhood of the other. This shows in particular that if  $\epsilon > 0$  is sufficiently small then  $\text{ev}(\overline{\mathcal{M}}_J^*)$  intersects the  $\epsilon$ -ball of normal coordinates since  $\text{ev}(\overline{\mathcal{M}}_{\hat{J}}^*)$  passes through its center. Second, we argue as above, using weighted Sobolev spaces  $W^{2,p}$  to show that the  $C^1$ -distance between the evaluation maps is  $\mathbf{O}(\epsilon)$ . Together with the above estimates on the Morse flow, it follows as in the argument at critical points above that  $(f, g, J)$  is generic with respect to rigid generalized disks and there is a unique generalized disk of  $(f, g, J)$  near every generalized rigid disk of  $(\hat{f}, \hat{g}, \hat{J})$ .

If  $\dim(L) = 2$  then an argument similar to that used for special points shows that it is possible to normalize at intersection points introducing only a small perturbation of rigid generalized disks.  $\square$

Consider a semi-normalized triple  $(\hat{f}, \hat{g}, \hat{J})$  as above and the corresponding 1-parameter family of Morse functions  $\hat{f}_\lambda = \lambda \hat{f}$ ,  $0 < \lambda \leq 1$ . We next deform the metric  $\hat{g}$  and the 1-parameter family of Morse functions  $\hat{f}_\lambda$ ,  $0 < \lambda \leq 1$  in a neighborhood of the rigid generalized disks defined by  $(\hat{f}, \hat{g}, \hat{J})$  in order to facilitate the construction of holomorphic disks from generalized disks, which will be carried out in Subsection 6.5. We will do this maintaining

genericity with respect to rigid generalized disks and without changing rigid generalized disks.

We say that  $(f_\lambda, g, J)$ , where  $f_\lambda$ ,  $0 < \lambda \leq 1$  is a 1-parameter family of Morse function, is *normalized* if  $(f_\lambda, g, J)$  is semi-normalized for each  $\lambda$  and if the following conditions hold:

- (n3) Outside a neighborhood of the flow lines of rigid generalized disks  $f_\lambda = \lambda f$  for some function  $f$ .
- (n4) Let  $\gamma$  be a flow line of a rigid generalized disk ending or beginning at a critical point  $p$ . Choose coordinates  $(x_1, \dots, x_n)$  around  $p$  as in (n2) such that  $\gamma$  corresponds to the  $x_1$ -axis. Then there is a *plateau point*  $q = (a, 0, \dots, 0)$  on  $\gamma$  (which lies between the critical point and any special point on  $\gamma$ ) such that

$$f_\lambda(q + y) = \begin{cases} \lambda(c + \sigma_1 a^2) + 2\sigma_1 a y_1 & \text{for } y_1 \geq 0, \\ \lambda(c + \sigma_1(a + y_1)^2 + \sum_{j=2}^n \sigma_j y_j^2) & \text{for } y_1 < -K\lambda, \end{cases}$$

where  $K > 0$  is a large constant.

Assume that there is a plateau point near every critical point and let  $\gamma$  be a flow line of a rigid generalized disk. Then parts of  $\gamma$  between special points, between special points and plateau points, or between plateau points are finite flow segments  $\hat{\gamma}$  with endpoints  $p_1$  and  $p_2$  around which there are coordinates  $(x_1, \dots, x_n) \in \mathbb{R}^n$  in which  $g$  is the standard Euclidean metric and in which function looks like  $f(x) = c_j + k_j x_1$ ,  $j = 1, 2$ . Pick two inflection points  $q_1$  and  $q_2$  between any two special points  $p_1$  and  $p_2$ . Assume that the order of these points along  $\gamma$  is  $p_1, q_1, q_2, p_2$ .

- (n5) In a neighborhood of the inflection points  $q_1$  and  $q_2$  there are coordinates  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$  such that the metric  $g$  is the flat metric  $g(u) = \sum_j du_j \otimes du_j$  and  $g(v) = \sum_j dv_j \otimes dv_j$ , and such that

$$f_\lambda(u) = \begin{cases} \lambda(c_1 + k_1 u_1), & \text{for } 0 \leq u_1 \leq K\lambda \\ \lambda(c_1 + k_1 u_1 - \alpha u_1^2), & \text{for } -2K\lambda \leq u_1 \leq -K\lambda, \end{cases}$$

$$f_\lambda(v) = \begin{cases} \lambda(c_2 + k_2 v_1 - \alpha v_1^2), & \text{for } 0 \leq u_1 \leq K\lambda \\ \lambda(c_2 + k_2 v_1), & \text{for } -2K\lambda \leq u_1 \leq -K\lambda, \end{cases}$$

where  $k_1$  and  $k_2$  are determined by the nearby special points as explained above and where  $\alpha = \frac{k_1^2 - k_2^2}{2(c_2 - c_1)}$ . We call the region  $-K\lambda \leq y_1 \leq 0$  the *inflection region*.

- (n6) There are flow coordinates  $x = (x_1, \dots, x_n) = (x_1, x')$  in a neighborhood  $U(\hat{\gamma})$  of  $\hat{\gamma}$  such that the following hold. The metric is the standard flat metric  $g(x) = \sum_j dx_j \otimes dx_j$ . The flow line  $\gamma$  corresponds to  $\{x: 0 \leq x_1 \leq T, x' = 0\}$  and outside a finite number of *inflection regions* of the form  $\{x: a \leq x_1 \leq a + K\lambda\}$  for some large constant  $K > 0$  the function  $f$  is given by

$$f_\lambda(x) = \lambda(c + \mu x_1 + \nu x_1^2),$$

for constants  $c, \mu, \nu$  where  $|\mu| + |\nu| \neq 0$ .

*Remark 6.7.* By (n3), outside the interpolation neighborhood, the functions  $f_\lambda$  are simply scalings of a function and we define a gradient line of  $f_\lambda$  in this region as a gradient line

of  $f$ . Inside the interpolation region the functions converge after rescaling to a piecewise linear Lagrangian. We define the Morse flow lines of  $f_\lambda$  as the flow lines of the rescaled limit function. We also point out that a normalized triple  $(f_\lambda, g, J)$  gives rise to a non  $C^1$  gradient field in the re-scaled limit obtained by scaling the fiber only. For local questions about holomorphic curves it is more useful to scale also the base and under such scaling the limit is smooth.

**Lemma 6.8.** *Let  $(\hat{f}, \hat{g}, \hat{J})$  be a semi-normalized triple and let  $\hat{f}_\lambda = \lambda \hat{f}$ ,  $0 < \lambda \leq 1$ . Then there exists a normalized triple  $(f_\lambda, g, J)$ ,  $0 < \lambda \leq 1$  such that (n3) hold with  $f = \hat{f}$ , with no generalized disks of formal dimension  $< 0$ , and with rigid generalized disks of dimension 0 which are identical to the rigid generalized disks of  $(\hat{f}, \hat{g}, \hat{J})$ .*

*Proof.* To prove this lemma we first note that a  $C^2$ -small deformation  $\tilde{f}_\lambda$  of  $\hat{f}_\lambda$  near critical points allows us to introduce plateau points and furthermore that this can be done without altering any flow line which is part of a rigid generalized disk. It then follows that the rigid generalized disks of  $(\tilde{f}_\lambda, \hat{g}, \hat{J})$  agree with those of  $(\hat{f}, \hat{g}, \hat{J})$ .

With plateau points introduced, we apply [3, Lemmas 4.5 and 4.6] to achieve (n5). More precisely, these lemmas allow us to find coordinates  $(u_1, \dots, u_n)$  around  $\hat{\gamma}$  which agree with the coordinates already defined at the endpoints of  $\hat{\gamma}$  and in which  $\tilde{f}_\lambda(u) = Q(u_1)$  where  $Q$  is a quadratic polynomial. The coordinates  $(u_1, \dots, u_n)$  are constructed from Fermi coordinates on level surfaces of  $\tilde{f}_\lambda$  perpendicular to  $\hat{\gamma}$  and hence the metric

$$g(u) = \sum_j du_j \otimes du_j$$

in a neighborhood of  $\hat{\gamma}$  is at  $C^1$ -distance  $\mathbf{O}(\eta)$  from  $\hat{g}$ , where  $\eta$  is the separation between special points, and of bounded  $C^2$ -distance from it provided the neighborhood is taken sufficiently small. Letting  $J$  be the almost complex structure induced by  $g$  we then find that we can take  $\hat{J}$  and  $J$  to be  $C^0$ -close as well; see Remark 6.1.

It remains to show that the rigid generalized disks agree with those of  $(\hat{f}, \hat{g}, \hat{J})$  and that there are no generalized disks of formal dimension  $< 0$  of  $(f_\lambda, g, J)$ . By construction, rigid generalized disks of  $(\hat{f}, \hat{g}, \hat{J})$  are rigid generalized disks of  $(f_\lambda, g, J)$  as well. As in the proof of Lemma 6.6, a  $C^0$ -small deformation of the almost complex structure leads to a  $C^0$ -small deformation of the evaluation map  $\text{ev}(\overline{\mathcal{M}^*})$ . In particular, for  $\hat{J}$  and  $J$  sufficiently close in  $C^0$ , any holomorphic disk part of a generalized rigid disk of  $(f_\lambda, g, J)$  lies in a small neighborhood of some holomorphic disk part of a rigid generalized disk of  $(\hat{f}, \hat{g}, \hat{J})$ . However, in such a neighborhood  $J = \hat{J}$  and it follows that the disks, and consequently the rigid generalized disks agree.

The same argument shows that there would exist generalized disks  $(\hat{f}, \hat{g}, \hat{J})$  of dimension  $< 0$  near any such generalized disk of  $(f_\lambda, g, J)$ . Since  $(\hat{f}, \hat{g}, \hat{J})$  is semi-normalized it is in particular generic with respect to rigid generalized disks and we conclude there are no generalized disks of dimension  $< 0$ .  $\square$

*Remark 6.9.* In the language of [3], the midpoints of the intervals where the functions  $f_\lambda$  in (n4) and (n5) are not necessarily given by a polynomial of degree at most two, are called edge points and their neighborhoods edge point regions. Although not mentioned explicitly in [3] also the  $\mathbf{O}(\lambda)$ -neighborhoods there should be taken to have diameter  $K\lambda$  where  $K$

is some large constant so that the Lagrangian is sufficiently close to the 0-section on the  $\lambda$ -scale. See Remark 6.12 for details.

*Remark 6.10.* The introduction of special points and coordinates along flow lines as described above essentially correspond to a piecewise linear approximation of the graph of  $df_\lambda$  with corners smoothed in regions of size  $K\lambda$ .

**6.4. From holomorphic to generalized disks.** Fix a Morse function  $f$ , a metric  $g$ , and an almost complex structure  $J$  such that the triple  $(f, g, J)$  is normalized as in Lemma 6.8. Let  $L_\lambda$  denote the Legendrian submanifold obtained by shifting  $L$  a large distance  $s$  upwards in the  $z$ -direction and then along  $f_\lambda$  (which is a function in a  $C^1 \mathbf{O}(\lambda)$ -neighborhood of the 0-function). With notation as in Lemma 2.5, we have  $\Pi_P(L_\lambda) = \Pi_P(L_1(f_\lambda))$ . As in Section 3.1, Reeb chords of  $L \cup L_\lambda$  are separated into pure chords, mixed chords, and Morse chords and we study  $J$ -holomorphic disks with boundary on  $L \cup L_\lambda$ . (Recall that a  $J$ -holomorphic disk with boundary on  $L \cup L_\lambda$  is a map  $u: D_{m+1} \rightarrow P$  such that  $u(\partial D_{m+1}) \subset \Pi_P(L \cup L_\lambda)$  and such that  $u|_{\partial D_{m+1}}$  has a continuous lift to  $L \cup L_\lambda$ ; see Subsection 2.2.3.) By Lemma 3.1, such a disk with one positive puncture has either zero or two mixed Reeb chords. Disks with all their punctures at pure Reeb chords correspond naturally to holomorphic disks with boundary on  $L$ . We therefore concentrate on disks with mixed punctures. In this subsection, we show that holomorphic disks with boundary on  $L \cup L_\lambda$  and with at least one Morse chord converge to generalized disks as  $\lambda \rightarrow 0$ . (Once the analysis of such disks is complete, disks without Morse chords are controlled by Lemmas 2.2 and 2.5; see the end of Section 6.5.)

In outline, the proof of generalized disk convergence runs as follows: For disks with two Morse chords, convergence follows from [3, Theorem 1.2] in combination with a monotonicity argument. For disks with one Morse chord, we represent the domains of the holomorphic maps  $u_\lambda$  with boundary on  $L \cup L_\lambda$  as strips  $\mathbb{R} \times [0, m]$ ,  $m \in \mathbb{Z}$ ,  $m \geq 1$  with slits around rays  $[a_j, \infty) \times \{j\}$ , where  $j \in \mathbb{Z}$ ,  $0 < j < m$ . After adding a uniformly finite number of punctures to the domains, we get a uniform derivative bound  $|du_\lambda| = \mathbf{O}(1)$ . It is then a consequence of Gromov compactness that this sequence converges to a broken disk  $u_0$  with boundary on  $L$ , uniformly on compact subsets. In the present situation, that does not give the full picture because areas of holomorphic disks with boundary on  $L \cup L_\lambda$  are not uniformly bounded from below as  $\lambda \rightarrow 0$ . For example, the holomorphic disks corresponding to a Morse flow line of  $f_\lambda$  constructed in [3, Theorem 1.3] have areas of size  $\mathbf{O}(\lambda)$  and on every compact neighborhood of a point which maps to a point in the flow line to which the disks converge, the holomorphic maps converge to a constant map. In order to capture the full picture, we must understand holomorphic disks with areas of size  $\mathbf{O}(\lambda)$  as well. To this end, we establish the existence of vertical line segments in the domain  $\Delta$  of the  $u_\lambda$  which subdivide  $\Delta$  into two pieces  $\Delta = \Delta_1 \cup \Delta_2$  such that on  $\Delta_2$ , the stronger derivative bound  $|du_\lambda| = \mathbf{O}(\lambda)$  holds. With this derivative bound established, the arguments from [3, Section 5] show that  $u_\lambda|_{\Delta_2}$  converges to a Morse flow line of  $f_\lambda$ . Finally, choosing “maximal”  $\Delta_2$ , we show that the limiting flow line of  $u_\lambda|_{\Delta_2}$  starts at a point on the boundary of a component of the (possibly broken) disk with boundary on  $L$  which is the limit of  $u_\lambda|_{\Delta_1}$ .

**Lemma 6.11.** *Let  $u_\lambda: D \rightarrow P$  be holomorphic disks with boundary on  $L \cup L_\lambda$  with the positive puncture and one negative puncture at Morse chords. Then  $u_\lambda$  has no other punctures*

and  $u_\lambda$  converges to a (possibly broken) Morse flow line of  $f_\lambda$  as  $\lambda \rightarrow 0$ . Furthermore, if the disks  $u_\lambda$  are rigid, then so is the limiting flow line.

*Proof.* By Lemma 2.3, the area of the disk is of size  $\mathbf{O}(\lambda)$ . By monotonicity, the disk cannot leave an  $\mathbf{O}(\lambda^{\frac{1}{2}})$ -neighborhood of  $\Pi_P(L)$ . Since the complex structure is standard in a such neighborhood of  $\Pi_P(L)$ , it follows from [3, Lemmas 5.13] that the holomorphic disk converges to a flow line. This flow line must furthermore be rigid by comparison of dimension formulas for holomorphic disks and flow lines, see [3, Proposition 3.18].  $\square$

*Remark 6.12.* In the proof of [3, Lemma 5.13], the edge point regions (see [3, Section 4.3.8], in the notation of this paper these regions are the regions around plateau points and inflection points described in (n4) and (n5) of Subsection 6.3, respectively) were not explicitly mentioned. For completeness, we give the explicit argument here. An edge point region is a region of size  $\mathbf{O}(\lambda)$  around a point on a flow line in a rigid generalized disk where the Lagrangian interpolates between its nearby affine pieces. In fact, these edge point regions should be chosen to have size  $K\lambda$ , where  $K$  is a sufficiently large constant. With such a choice, the derivative of the interpolation function can be taken as small as  $\mathbf{O}(\frac{1}{K})$  after rescaling of base and fiber by  $\lambda^{-1}$  as compared to all other gradient differences nearby. In the proof of the convergence result [3, Lemma 5.12], one uses a split coordinate system near edge points. In the directions perpendicular to the flow line the argument is the one given in the lemma. In the directions along the flow line the re-scaled correction function  $f$  in [3, Equation (5-8)] is now  $\mathbf{O}(\frac{1}{K})$  rather than  $\mathbf{O}(\lambda)$ . However, changes in this direction just correspond to reparametrization of the holomorphic disk (i.e. variations along the flow direction). The fact that the error term is small after rescaling shows that the time spent by a holomorphic map  $u_\lambda$  in the  $K\lambda$ -neighborhood (the length of the part of the domain mapping to the  $K\lambda$ -neighborhood) is  $\mathbf{O}(1)$ . Consequently, the disk lies at most  $\mathbf{O}(\lambda)$  from a flow line after having passed through the edge point region.

We next work with disks that have one mixed Morse puncture and another mixed puncture. It will be convenient to think of the domains and spaces of conformal structures of our holomorphic disks as “standard domains” in the language of [3]. For details we refer to [3, Subsection 2.2.1]; here we give a brief description. Consider  $\mathbb{R}^{m-2}$  with coordinates  $a = (a_1, \dots, a_{m-2})$ . Let  $t \in \mathbb{R}$  act on  $\mathbb{R}^{m-2}$  by  $t \cdot a = (a_1 + t, \dots, a_{m-2} + t)$ . The orbit space of this action is  $\mathbb{R}^{m-3}$ . Define a *standard domain*  $\Delta_m(a)$  as the subset of  $\mathbb{R} \times [0, m]$  obtained by removing  $m-2$  horizontal slits of width  $\epsilon$ ,  $0 < \epsilon \ll 1$ , starting at  $(a_j, j)$ ,  $j = 1, \dots, m-2$  and going to  $\infty$ . All slits have the same shape, ending in a half-circle, see Figure 10.

Endowing  $\Delta_m(a)$  with the flat metric, we get a conformal structure  $\kappa(a)$  on the  $m$ -punctured disk. Moreover, using the fact that translations are biholomorphic, we find that  $\kappa(a) = \kappa(t \cdot a)$  for all  $a$ . As shown in [3, Lemma 2.2],  $\kappa$  is a diffeomorphism from  $\mathbb{R}^{m-2}$  to the space of conformal structures on the  $m$ -punctured disk. Below we will often drop  $a$  from the notation and write  $\Delta_m$  for a standard domain.

If  $I$  is a boundary component of  $\Delta_m[a]$  which has both of its ends at  $\infty$ , we will call the point with smallest real part along  $I$  a *boundary minimum* of  $\Delta_m$ . A *vertical line segment* of a standard domain  $\Delta_m$  is a line segment of the form  $\{\tau\} \times [a, b]$  contained in  $\Delta_m$  and with  $(\tau, a)$  and  $(\tau, b)$  in  $\partial\Delta_m$ .

The first step toward establishing generalized disk convergence is to show that for any sequence of  $J$ -holomorphic disks  $u_\lambda: \Delta_m^\lambda \rightarrow P$  with boundary on  $L \cup L_\lambda$ , where  $\Delta_m^\lambda$  are

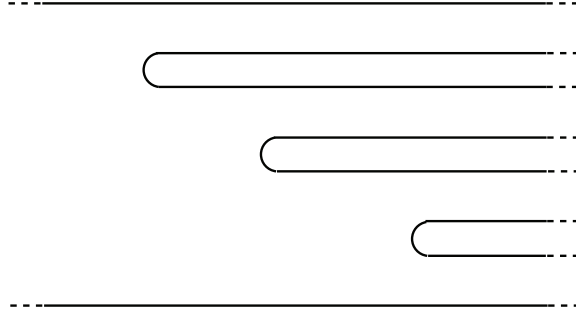


FIGURE 10. A standard domain

standard domains, there are neighborhoods of each Morse puncture where the disks converge to a (possibly constant) flow line.

The key to establishing this convergence is a certain derivative bound. More precisely, we have the following. As in Subsection 2.1 let  $\theta$  denote the 1-form on  $P$  which is the primitive of its symplectic form. Fix a constant  $M > 0$  and let  $l_\lambda \approx [0, 1]$  be a vertical segment in  $\Delta_m^\lambda$  with the following properties:

- (11)  $u_\lambda(l_\lambda)$  is contained in an  $\mathbf{O}(\lambda)$ -neighborhood of  $L$ .
- (12)  $\left| \int_{l_\lambda} u_\lambda^*(\theta) \right| \leq M\lambda$ .
- (13)  $||z(1) - z(0)| - s| \leq M\lambda$ , where we think of 0 and 1 as the endpoints of  $l_\lambda$  and  $z$  as the  $\mathbb{R}$  coordinate in  $P \times \mathbb{R}$  (evaluated on the boundary lift of  $u$ ).

Note that  $l_\lambda$  subdivides  $\Delta_m^\lambda$  into two components. Let  $\Delta^\lambda(l_\lambda)$  denote the component which contains the Morse puncture. Also, for  $d > 0$ , let  $\Delta^\lambda(l_\lambda, d)$  denote the subset of points in  $\Delta^\lambda(l_\lambda)$  which are at distance at least  $d$  from  $l_\lambda$ .

**Lemma 6.13.** *For all sufficiently small  $\lambda > 0$ , the following derivative bound holds:*

$$|du_\lambda(z)| = \mathbf{O}(\lambda), \quad z \in \Delta^\lambda(l_\lambda, 1).$$

*Proof.* The idea of the proof is to localize the situation and then use technology from the analysis of flow lines (trees) in [3, Section 5].

Let  $b$  denote the Morse chord. We first show that the area of  $u_\lambda(\Delta^\lambda(l_\lambda))$  is of order  $\mathbf{O}(\lambda)$ . If the puncture of  $u_\lambda$  at  $b$  is negative, then the area of  $u_\lambda(\Delta^\lambda(l_\lambda))$  is smaller than

$$\int_{l_\lambda} u_\lambda^*(\theta) + |z(1) - z(0)| - \ell(b) - \sum \ell(c),$$

where the sum ranges over all negative punctures in  $\Delta^\lambda(l_\lambda)$ . Since the second and third terms are both of size  $s + \mathbf{O}(\lambda)$ , since the first term is of size  $\mathbf{O}(\lambda)$ , and since  $\ell(c) > \ell_{\min} > 0$  for all Reeb chords  $c$ , we find that the sum must be empty, and consequently that the area of  $u_\lambda(\Delta^\lambda(l_\lambda))$  is of size  $\mathbf{O}(\lambda)$ .

If the puncture of  $u_\lambda$  at  $b$  is positive, the area of  $u_\lambda(\Delta^\lambda(l_\lambda))$  is smaller than

$$\ell(b) - \int_{l_\lambda} u_\lambda^*(\theta) - |z(1) - z(0)| - \sum \ell(c),$$

and an analogous argument shows that there are no negative punctures and that the area of  $u_\lambda(\Delta^\lambda(l_\lambda))$  is also of size  $\mathbf{O}(\lambda)$ .

We conclude by monotonicity that  $u_\lambda(\Delta^\lambda(l_\lambda))$  must lie in an  $\mathbf{O}(\lambda^{\frac{1}{2}})$ -neighborhood of  $L$ . Since  $J$  agrees with the almost complex structure induced by the metric on  $L$  in such a neighborhood and since, in local coordinates given by a symplectic neighborhood map  $\Phi: T^*L \rightarrow X$ ,  $\theta = p dq$ , [3, Lemma 5.4] shows that the function  $|p|^2$ , where  $p$  is the fiber coordinate in  $T^*L$  composed with  $u_\lambda$ , is subharmonic on  $\Delta^\lambda(l_\lambda)$  and therefore attains its maximum on the boundary. The lemma then follows from [3, Lemma 5.6].  $\square$

**Corollary 6.14.** *The restrictions  $u_\lambda|_{\Delta^\lambda(l_\lambda, \log(\lambda^{-1}))}$  converge to a flow line of  $f_\lambda$ . (Here we also allow constant flow lines.)*

*Proof.* Note that any region of diameter  $\log(\lambda^{-1})$  maps inside a disk of radius  $\mathbf{O}(\lambda \log(\lambda^{-1}))$ . Moreover, along any strip region in  $\Delta^\lambda(l_\lambda, \log(\lambda^{-1}))$ , the maps converge to a flow line at rate  $\mathbf{O}(\lambda)$  by the proof of [3, Theorem 1.2].  $\square$

*Remark 6.15.* If the limiting flow line in Corollary 6.14 is constant, then it lies at  $\Pi_P(c)$  for some Reeb chord  $c$  of  $L$ . To see this, note that  $\Delta^\lambda(l_\lambda, \log(\lambda^{-1}))$  always contains a half infinite strip and that if the starting point of this strip does not converge to the projection of a Reeb chord, then the flow line is non-constant.

6.4.1. *Blow up analysis.* We next show that for any sequence of  $J$ -holomorphic disks  $u_\lambda$  with boundary on  $L \cup L_\lambda$ , we can choose conformal representatives  $\Delta_m^\lambda$  of their domains such that the derivatives  $|du_\lambda|$  are uniformly bounded. When a bubble forms in a sequence of maps on such domains, some coordinate of the domains  $\Delta_m^\lambda$  in the space of conformal structures goes to  $\infty$  (rather than that the derivative of  $u_\lambda$  blowing up).

**Lemma 6.16.** *Let  $u_\lambda: \Delta_m^\lambda \rightarrow P$  be a sequence of  $J$ -holomorphic disks with boundary on  $L \cup L_\lambda$ . After addition of a finite number of punctures in  $\Delta_m^\lambda$ , creating new domains  $\Delta_{m+k}^\lambda$ , the induced maps  $u_\lambda: \Delta_{m+k}^\lambda \rightarrow P$  satisfy a uniform derivative bound.*

*Proof.* The proof is a standard blow up argument, so we only sketch the details. Assume that  $M_\lambda = \sup_{\Delta_m^\lambda} |du_\lambda|$  is not bounded. The asymptotics near the punctures of  $u_\lambda$  show that there exist points  $p_\lambda \in \Delta_m^\lambda$  at which  $|du_\lambda| = M_\lambda$ . Consider the sequence of maps  $g_\lambda = u_\lambda \left( p_\lambda + \frac{z}{M_\lambda} \right)$  defined on  $\{z \in \mathbb{C} : (p_\lambda + \frac{z}{M_\lambda}) \in \Delta_m^\lambda\}$ . Note that the derivatives of these maps are uniformly bounded. Therefore we can extract a convergent subsequence. This gives a non-constant holomorphic disk with boundary on  $L$  which has one positive puncture and no other puncture. Denote this limit disk  $v^{[1]}: D \rightarrow P$  and fix a local hypersurface  $H$  transversely intersecting  $v^{[1]}(\partial D)$  at a point far from all Reeb chords. It follows from the convergence  $g_\lambda \rightarrow v^{[1]}$  that there exists a point in a neighborhood of  $p_\lambda$  which  $u_\lambda$  maps to  $H$ . Puncturing  $\Delta_m^\lambda$  at this point induces a new sequence of maps  $u_\lambda^{[1]}: \Delta_{m+1}^\lambda \rightarrow P$ . If  $|du_\lambda^{[1]}|$  is uniformly bounded then the lemma follows.

Assume that  $\sup_{\Delta_{m+1}^\lambda} |du_\lambda^{[1]}|$  is unbounded. Arguing as above, we find another bubble  $v^{[2]}$  in the limit with one positive puncture and no other punctures. Adding another puncture in the domain which corresponds to some point in  $v^{[2]}$  in the limit, we get new maps and domains  $u_\lambda^{[2]}: \Delta_{m+2}^\lambda \rightarrow P$ . Repeating this, we either have no derivative blow up in which

case the lemma follows or we add punctures as above. To see that this is a finite process note that each bubble has area bounded from below by the length of the shortest Reeb chord and that the sum of the areas of all bubbles must be smaller than the length of the longest Reeb chord of  $L$ .  $\square$

*Remark 6.17.* Let  $p$  be the image point corresponding to an additional puncture added in the procedure described above. Note that there exists a disk of finite radius around  $p$  which does not contain any Reeb chords. It follows from monotonicity that for  $\lambda > 0$  small enough, the area contribution of any subdisk obtained by cutting off a vertical segment which connects  $L$  to  $L_\lambda$  and which contains the puncture corresponding to the marked point is uniformly bounded from below. It follows that  $\Delta^\lambda(l_\lambda)$  in Lemma 6.13 cannot contain any additional punctures.

**6.4.2. Generalized disk convergence.** Consider a sequence  $u_\lambda$  of  $J$ -holomorphic disks with boundary on  $L \cup L_\lambda$  with one puncture at a Morse chord. Using Lemma 6.16, we assume that these are maps  $u_\lambda: \Delta_m^\lambda \rightarrow P$  with uniformly bounded derivatives. It is a consequence of Gromov compactness that this sequence converges to a broken disk  $v$  on  $L$ , uniformly on compact subsets. Let  $\partial v$  denote the image in  $L$  of the boundary of the possibly broken non-constant limit disk  $v$  or if there are only constant limit disks then let  $\partial v$  denote the double point corresponding to the mixed puncture of the disk which is not a Morse chord. Lemma 6.14 and Remark 6.17 imply that if a vertical line segment  $l_\lambda \subset \Delta_m^\lambda$  satisfies (11) – (13) then on  $\Delta_m^\lambda(l_\lambda)$  the disks converge to a flow line.

**Lemma 6.18.** *There exist vertical segments  $l_\lambda$  which satisfies (11) – (13) and such that  $u_\lambda(l_\lambda)$  converges to a point in  $\partial v$ .*

*Proof.* We prove this lemma by contradiction: if the statement of the lemma does not hold, then the area difference between a limit disk and the disks before the limit violates an  $\mathbf{O}(\lambda)$  bound derived from Stokes' theorem.

Thus, we assume that the lemma does not hold. Then there exists  $\epsilon > 0$  such that for any sequence of  $l_\lambda$  which satisfies (11) – (13), some point on  $l_\lambda$  maps a distance at least  $\epsilon > 0$  from  $\partial v$ . Consider a strip region  $[-d, d] \times [0, 1] \subset \Delta_m^\lambda$  for which some point converges to a point a distance  $\delta$  from  $\partial v$ , where  $\frac{\epsilon}{4} < \delta < \frac{\epsilon}{2}$ . Let  $\sup_{[-d, d] \times [0, 1]} |du_\lambda| = K$ . Then  $K$  is not bounded by  $M\lambda$  for any  $M > 0$ . Since the difference between the area of the limit disk  $v$  and that of  $u_\lambda$  is of order of magnitude  $\mathbf{O}(\lambda)$  it follows that  $|du_\lambda| = \mathbf{O}(\lambda^{\frac{1}{2}})$  from the usual bootstrap estimate. Thus  $K = \mathbf{O}(\lambda^{\frac{1}{2}})$ . Consider next the scaling of the target by  $K^{-1}$ . We get a sequence of maps  $\hat{u}_\lambda$  from  $[-d, d] \times [0, 1]$  with bounded derivative. Note moreover that the boundary condition is  $\mathbf{O}(\lambda^{\frac{1}{2}})$  from the 0-section. Changing coordinates to the standard  $(\mathbb{C}^n, \mathbb{R}^n)$  respecting the complex structure at the limit point, we find that there are maps  $f_\lambda: [-d, d] \times [0, 1] \rightarrow \mathbb{C}^n$  with the following properties

- $\sup_{[-d, d] \times [0, 1]} |D^k f_\lambda| = \mathbf{O}(\lambda^{\frac{1}{2}})$ ,  $k = 0, 1$ .
- $u_\lambda + f_\lambda$  satisfies  $\mathbb{R}^n$  boundary conditions
- $\bar{\partial}(u_\lambda + f_\lambda) = \mathbf{O}(\lambda^{\frac{1}{2}})$ .

It follows that  $u_\lambda + f_\lambda$  converges to a holomorphic map with boundary on  $\mathbb{R}^n$ , which takes 0 to 0 and which has derivative of magnitude 1 at 0. Using solubility of the  $\bar{\partial}$ -equation in combination with  $L^2$ -estimates in terms of area we find that the area of  $\hat{u}_\lambda$  must be

uniformly bounded from below by a constant  $C$ . The area contribution to the original disks near the limit is thus at least  $K^2 C$ . Since  $[-d, d] \times [0, 1]$  covers a length along  $L$  of at most  $2Kd$ , we may repeat the argument with many disjoint finite strips which together cover a finite length and with maximal derivatives  $K_j$ . We find that the area contribution is bounded from below by  $C \sum K_j^2$  and that, since the length contribution is finite, we get:

$$2d \sum K_j \geq \frac{\epsilon}{100}.$$

Now,

$$C \sum K_j^2 \geq C \inf_j \{K_j\} \sum K_j \geq C' \inf_j \{K_j\}.$$

For any  $M > 0$ ,  $\inf_j \{K_j\} \geq M\lambda$ . To see this assume that it does not hold true. Then there is a sequence of vertical segments  $l_\lambda$  such that  $|du_\lambda| \leq 2M\lambda$  with the property that the distance between  $u(l_\lambda)$  and  $\partial v$  is at most  $\frac{3}{4}\epsilon$ . This however contradicts our hypothesis. Consequently, the area contribution from the remaining part of the disk is not  $\mathbf{O}(\lambda)$ , which contradicts Stokes' theorem.  $\square$

As a consequence we get the following:

**Corollary 6.19.** *Any sequence of rigid holomorphic disks  $u_\lambda: \Delta_m^\lambda \rightarrow P$  with boundary on  $L \cup L_\lambda$  and with one Morse puncture has a subsequence which converges to a rigid generalized disk.*

*Proof.* It follows from Lemmas 6.14 and 6.18 and from Gromov compactness that the limit gives a generalized disk. This generalized disk must furthermore have formal dimension 0. Since the triple  $(f_\lambda, g, J)$  is normalized, it is in particular generic with respect to rigid generalized disks, see Lemma 6.8, and it follows that the  $J$ -holomorphic disk part is not broken and that the generalized disk is transversely cut out.  $\square$

**6.5. From generalized to holomorphic disks.** Let  $f_\lambda$ ,  $0 < \lambda \leq 1$  be a 1-parameter family of Morse functions on  $L$ ,  $g$  be a metric on  $L$ , and  $J$  an almost complex structure on  $P$  such that the triple  $(f_\lambda, g, J)$  is normalized. The goal of this section is to produce a unique rigid holomorphic disk with boundary on  $L \cup L_\lambda$  near each generalized holomorphic disk determined by  $(f_\lambda, g, J)$  for all sufficiently small  $\lambda > 0$ . We begin by associating to each generalized holomorphic disk a family of domains and approximately holomorphic maps with boundary on  $L \cup L_\lambda$ . We then define a functional analytic space of variations of the approximately holomorphic map, show that the linearized  $\bar{\partial}_J$ -operator is uniformly invertible on this space, and derive a second derivative estimate. By Lemma 6.2, the invertibility and second derivative estimate gives a unique holomorphic disk in the functional analytic neighborhood of the approximate solution, and we show that any solution must lie in this neighborhood.

**6.5.1. Approximate solutions.** Consider a rigid generalized disk  $(u, \gamma)$ , where  $u: \Delta_{m-1} \rightarrow P$  is the holomorphic disk part with  $m-1$  punctures mapping to Reeb chords and  $\gamma$  is the Morse flow line part. We consider the following three cases separately:

- (gd1) The map  $u$  is constant,
- (gd2) The map  $u$  is non-constant and  $\gamma$  is constant, and
- (gd3) The map  $u$  is non-constant and  $\gamma$  is non-constant.

For  $(u, \gamma)$  of type **(gd1)**, the rigid generalized disk is a flow line and existence of a unique holomorphic disk with boundary on  $L \cup L_\lambda$  near the rigid generalized disk follows from [3, Theorem 1.3].

In cases **(gd2)** and **(gd3)**, add a puncture to  $u$  at the junction point to obtain a map  $u: \Delta_m \rightarrow P$ . Fix a reference point 0 in  $\Delta_m$  which  $u$  maps to a point far from any puncture and note that the derivative of  $u$  is bounded. It follows from the standard form of the Lagrangian projection and the complex structure near junction points in **(n2)** that, in a half strip neighborhood of the junction point  $[0, \infty) \times \mathbb{R}$ , the map  $u$  looks like:

$$(6.9) \quad u(z) = \sum_{n < 0} c_n e^{n\pi(z)}, \quad z \in [0, \infty) \times \mathbb{R}, \quad c_n \in \mathbb{R}^n,$$

where  $\mathbb{R}^n$  corresponds to  $L$ . Similarly, near a Reeb chord puncture as in **(a2)**, the map  $u = (u_1, \dots, u_n)$  looks like:

$$(6.10) \quad u_j(z) = \sum_{n \leq 0} c_{j,n} e^{(-\theta_j + n\pi)z}, \quad z \in [0, \infty) \times [0, 1], \quad c_{j,n} \in \mathbb{R},$$

where  $0 < \theta_j < \pi$ . Consequently, there are vertical segments  $l_\lambda$  in a half strip neighborhood of any puncture at distance  $\mathbf{O}(\log(\lambda^{-1}))$  from  $0 \in \Delta_m$  such that a finite neighborhood of  $l_\lambda$  maps into an  $\mathbf{O}(\lambda)$  neighborhood of the corresponding special point.

In case **(gd2)**, we construct approximately holomorphic maps  $w_\lambda: \Delta_m \rightarrow P$  with boundary on  $L \cup L_\lambda$  as follows. At each Reeb chord puncture of  $u$  and at the junction point consider vertical line segments  $l_\lambda$  in  $\Delta_m$ . The vertical line segment  $l_\lambda$  subdivides  $\Delta_m$  into an inner component  $\Delta_m^0(\lambda)$  containing 0 and outer half strip regions near punctures. We change the boundary condition of  $u$  on  $\Delta_m^0(\lambda)$  according to the boundary lift of the rigid disk. To this end we must move the boundary of  $u|_{\Delta_m^0(\lambda)}$  a distance  $\mathbf{O}(\lambda)$ . Supporting such a deformation in an  $\mathbf{O}(\lambda^{\frac{1}{2}})$ -neighborhood of the boundary, one can achieve this while changing the derivative of  $u$  by at most  $\mathbf{O}(\lambda^{\frac{1}{2}})$ . Finally, we interpolate to constant maps to double points in finite region in  $\Delta_m$  near the vertical segments  $l_\lambda$ . Then the derivative of the resulting map in these finite regions is of size  $\mathbf{O}(\lambda)$ . This completes the definition of the approximately holomorphic maps  $w_\lambda: \Delta_m \rightarrow P$  in case **(gd2)**.

In case **(gd3)**, the construction of approximately holomorphic maps  $w_\lambda: \Delta_m \rightarrow P$  is a bit more involved. Here we subdivide the domain into three pieces. First pick vertical segments  $l_\lambda$  in  $\Delta_m$  near each Reeb chord puncture exactly as in case **(gd2)**. Near the junction point puncture we instead pick a vertical segment  $l$  which is mapped into a small but finite  $\epsilon$ -neighborhood of the junction point. Such a segment lies at distance  $\mathbf{O}(1)$  from  $0 \in \Delta_m$ . Cut  $\Delta_m$  off at  $l$  and  $l_\lambda$  and denote the component containing 0 by  $\Delta_m^0(\lambda)$ . The two other pieces are half strips  $[0, d\lambda^{-1}] \times [0, 1]$ , where  $d > 0$  is a suitable constant and a half infinite strip  $[0, \infty) \times [0, 1]$ . To construct  $\Delta_m$ , we glue  $[0, d\lambda^{-1}] \times [0, 1]$  to  $\Delta_m^0(\lambda)$  along  $l$  and glue in  $[0, \infty) \times [0, 1]$  to complete the resulting domain.

On the piece  $\Delta_m^0(\lambda)$ , we define the almost holomorphic map  $w_\lambda^0: \Delta_m^0 \rightarrow P$  by moving the boundary condition of  $u$  to  $L \cup L_\lambda$  according to the boundary lift of the rigid generalized disk and then interpolate to constant maps at all Reeb chord punctures, exactly as in case **(gd2)**.

Near the Morse flow line  $\gamma$ , cut off at a small distance from the junction point. As in [3, Section 6.1], we construct almost holomorphic maps  $w_\lambda^\gamma: [0, \infty) \times [0, 1] \rightarrow P$  which agree

with the natural holomorphic strip over the gradient flow lines in the flat metric, as in [3, Sections 6.1.1 and 6.1.2] outside the regions near plateau- and inflection points of **(n4)** and **(n5)**, where it interpolates between these solutions and satisfies  $|\bar{\partial}_J w_\lambda^\gamma| = \mathbf{O}(\lambda)$ .

Finally, consider the region near the junction point. We define a holomorphic map  $w_\lambda^{\text{jun}}: [0, d\lambda^{-1}] \times [0, 1] \rightarrow P$  which satisfies the boundary conditions. To define this map, we assume that the  $\mathbb{C}^n$ -coordinates  $x + iy$  have been chosen so that  $L$  corresponds to  $\mathbb{R}^n = \{y_1 = \dots = y_n = 0\}$  and  $L_\lambda$  corresponds to  $\{y_1 = \lambda, y_2 = \dots = y_n = 0\}$ . The holomorphic map is then

$$(6.11) \quad w_\lambda^{\text{jun}}(z) = (\lambda z, 0, \dots, 0) + \sum_{n < 0} c_n e^{n\pi z}, \quad c_n \in \mathbb{R}^n.$$

Near the junction point, the maps  $w_\lambda^0$  and  $w_\lambda^{\text{jun}}$  (as well as  $w_\lambda^\gamma$  and  $w_\lambda^{\text{jun}}$ ) are then of distance  $\mathbf{O}(\lambda)$  apart (see (6.9) and (6.11)), and we interpolate between them using a function of size  $\mathbf{O}(\lambda)$  on a finite rectangle. The function resulting from this interpolation is  $w_\lambda: \Delta_m \rightarrow P$  in case **(gd3)**.

As mentioned above, we will produce  $J$ -holomorphic disks near  $w_\lambda: \Delta_m \rightarrow P$  parametrizing a neighborhood of this map by a weighted Sobolev space of vector fields. We next describe the weight function in cases **(gd2)** and **(gd3)**. In the former case, this is straightforward: take  $h: \Delta_m \rightarrow \mathbb{R}$  to be a function which equals 1 on  $\Delta_m(\lambda_0)$  for some fixed  $\lambda_0$ . On the remaining strip regions of the form  $[0, \infty) \times [0, 1]$  we take  $h(\tau + it) = e^{\delta|\tau|}$  for  $\delta > 0$ , where  $\delta \ll \theta_j$  for all  $\theta_j$  as in (6.10) at any Reeb chord. In the latter case, the function  $h: \Delta_m \rightarrow \mathbb{R}$  equals 1 on the piece  $\Delta_m^0(\lambda_0)$  for some fixed  $\lambda_0$  and equals to  $e^{\delta|\tau|}$  for  $\tau + it \in [0, \infty) \times [0, 1]$  in the neighborhood  $[0, \infty) \times [0, 1]$  of each Reeb chord puncture for small  $\delta > 0$ . In  $[0, \frac{d}{\lambda}] \times [0, 1] \subset \Delta_m$ , we let:

$$(6.12) \quad h(\tau + it) = e^{\delta\left(\frac{1}{2}d\lambda^{-1} - \left|\tau - \frac{1}{2}d\lambda^{-1}\right|\right)}.$$

In  $[0, \infty) \times [0, 1]$ , the function  $h$  has the same shape as in (6.12) along edges where  $w_\lambda^\gamma$  agrees with an explicit solution and equals 1 in a neighborhood of the interpolation regions at plateau- and inflection points, as in [3, Section 6.3.1].

Let  $\|\cdot\|_{k,\delta}$  denote the Sobolev norm in the Sobolev space of functions with  $k$  derivatives in  $L^2$  with weight  $h$ .

**Lemma 6.20.** *The approximately holomorphic function  $w_\lambda$  satisfies*

$$\|\bar{\partial}_J w_\lambda\|_{1,\delta} = \mathbf{O}(\lambda^{\frac{3}{4}-\delta}(\log \lambda^{-1})^{\frac{1}{2}})$$

*Proof.* The regions where the map is non-holomorphic are of two kinds: finite rectangles where the size of the derivatives are  $\mathbf{O}(\lambda)$  and an  $\mathbf{O}(\lambda^{\frac{1}{2}})$ -neighborhood of the boundary in  $\Delta_m(d)$  cut off at all  $l_\lambda^{\text{Reeb}}$ . The former regions give a contribution of size  $\mathbf{O}(\lambda)$  the latter gives the contribution

$$\sqrt{\mathbf{O}(\lambda) \cdot \mathbf{O}(\lambda^{-2\delta}) \cdot \mathbf{O}(\lambda^{\frac{1}{2}}) \cdot \mathbf{O}(\log \lambda^{-1})},$$

where the first factor is the square of the size of the deformation, the second factor is the maximum value of the weight function  $h$  at a point where  $w_\lambda$  is not holomorphic, and the product of the last two estimates the area of the region in which  $w_\lambda$  is non-holomorphic.  $\square$

We associate a variation space  $\hat{\mathcal{H}}_{2,\delta}$  to  $w_\lambda$ . This is a direct sum

$$\hat{\mathcal{H}}_{2,\delta} = \mathcal{H}_{2,\delta} \oplus V_{\text{con}} \oplus V_{\text{sol}},$$

where the summands are the following:

- In both cases **(gd2)** and **(gd3)**,  $\mathcal{H}_{2,\delta}$  is a Sobolev space of vector fields  $v$  along  $w_\lambda$  with two derivatives in  $L^2$  weighted by  $h$  which satisfy the following additional conditions:  $v$  is tangent to  $L$  along the boundary,  $\bar{\nabla}_J v = 0$  along the boundary (as in [8, Lemma 3.2] and [3, Section 6.3.1]). In case **(gd3)**, the vector fields  $v$  satisfy the following additional conditions:  $v$  vanishes at one boundary point midway between plateau-points and inflection points, midway between inflection points (as in [3, Section 6.3.1]) as well as at a boundary point in the middle of the strip at the junction point. In other words,  $v$  vanishes at one of the boundary points of every vertical segment along which the weight function  $h$  has a local maximum.
- In both cases **(gd2)** and **(gd3)**,  $V_{\text{con}}$  is the space of conformal variations of  $\Delta_m$  (see [5, Section 5.6] for a description).
- In case **(gd2)**,  $V_{\text{sol}} = 0$ . In case **(gd3)**,  $V_{\text{sol}}$  is a finite dimensional space consisting of cut off constant solutions of the  $\bar{\partial}_J$ -equation supported in the regions where the weight function  $h$  is large, as follows. Along the flow line part, there are  $n$ -dimensional cut-off constant solutions between any two plateau/inflection-points exactly as in [3, Section 6.3.2]. There is also an  $n$ -dimensional space corresponding to the junction point. More precisely, in the coordinates of (6.11), this  $n$ -dimensional space is spanned by

$$\beta(1, 0, \dots, 0), \beta(0, 1, 0, \dots, 0), \dots, \beta(0, \dots, 0, 1),$$

where  $\beta$  is a cut off function equal to 1 in the region  $[0, d\lambda^{-1}] \times [0, 1]$  corresponding to the junction point and equal to 0 outside a uniformly finite neighborhood of it.

We equip  $V_{\text{sol}}$  with the supremum norm.

Choosing a 1-parameter family of Riemannian metrics  $G^\sigma$ ,  $0 \leq \sigma \leq 1$ , on  $P$  as in [8, Section 3.1.2], see also [5, Section 5.2], and an extension  $z: \Delta_m \rightarrow \mathbb{R}$  of the  $z$ -coordinate of the boundary lift of  $w_\lambda$ , we define an exponential map:

$$\exp(v) = \exp_{w_\lambda(\zeta)}^{G^\sigma(z(\zeta))}(v(\zeta)),$$

where  $\sigma: \mathbb{R} \rightarrow [0, 1]$ , and where  $\exp^{G^\sigma}$  is the exponential map in the metric  $G^\sigma$ , which gives a local chart in the configuration space of maps around  $w_\lambda$ . In particular, as in Subsection 6.1.6, we think of the (non-linear)  $\bar{\partial}_J$ -operator on  $\hat{\mathcal{H}}_{2,\delta}$  as the operator

$$\bar{\partial}_J(v) = d\exp(v) + Jd\exp(v)j.$$

**6.5.2. Uniform invertibility.** Let  $L\bar{\partial}_J$  denote the linearization of the  $\bar{\partial}_J$ -operator acting on elements  $v \in \hat{\mathcal{H}}_{2,\delta}$ . This map takes vector fields in  $\hat{\mathcal{H}}_{2,\delta}$  to complex anti-linear maps  $T\Delta_m \rightarrow w_\lambda^*TP$ . We pick a trivialization of  $T\Delta_m$  and identify the complex anti-linear map with the image of the trivializing vector field. In this way, we view the  $L\bar{\partial}_J$  as a map  $\hat{\mathcal{H}}_{2,\delta} \rightarrow \mathcal{H}_{1,\delta}$  where  $\mathcal{H}_{1,\delta}$  is the Sobolev space of vector fields along  $w_\lambda$  with one derivative in  $L^2$  weighted by  $h$ .

**Lemma 6.21.** *The differential*

$$L\bar{\partial}_J: \hat{\mathcal{H}}_{2,\delta} \rightarrow \mathcal{H}_{1,\delta}$$

*is uniformly invertible.*

*Proof.* The proof in case **(gd3)** is similar to the proof of [3, Proposition 6.20], so we just give an outline. Recall that the domain of  $w_\lambda$  was built out of three pieces  $\Delta_m = \Delta_m^0 \cup [0, d\lambda^{-1}] \times [0, 1] \cup [0, \infty) \times [0, 1]$ . Consider a variation  $v_\lambda$  of  $w_\lambda$ . Write  $v_\lambda = v_\lambda^0 + v_\lambda^{\text{jun}} + v_\lambda^{\text{mo}}$  where we use cut off functions to subdivide  $v$  into a disk-piece  $v^0$  supported in a neighborhood of  $\Delta_m^0$ , a junction-piece  $v^{\text{jun}}$  supported in a neighborhood of  $[0, d\lambda^{-1}] \times [0, 1]$ , and a Morse-piece  $v^{\text{mo}}$  supported in a neighborhood of  $[0, \infty) \times [0, 1]$ . If the operator is not uniformly invertible, then there is a sequence of variation maps  $v_\lambda$  such that

$$(6.13) \quad \|v_\lambda\|_{2,\delta} = 1, \quad \|L\bar{\partial}_J v_\lambda\|_{1,\delta} \rightarrow 0.$$

Parametrize a neighborhood of the holomorphic disk part using a Sobolev space with small positive exponential weight near the added marked point and cut off constant solutions transverse to the disk. We infer from the properties on the disk part that  $v_\lambda^0$  has non-zero component along the cut-off and conformal solutions at the junction point. Similarly,  $v_\lambda^{\text{mo}}$  has non-zero component in the tangent directions of the (un)stable manifold in which it lies. The components of the cut-off solutions correspond to cut-off solutions at the junction point in the space of variations of  $w_\lambda$ . Since the evaluation map from the moduli space of holomorphic disks is transverse to the (un)stable manifold at the junction point, it follows that the component of  $v_\lambda^{\text{jun}}$  along the cut off solutions must go to 0 with  $\lambda$ . We conclude from this that  $v_\lambda^0 \rightarrow 0$  and  $v_\lambda^{\text{mo}} \rightarrow 0$ . Consider the affine inclusion  $[0, d\lambda^{-1}] \times [0, 1] \subset \mathbb{R} \times [0, 1]$  taking  $\frac{d}{2\lambda}$  to 0. Since the  $\bar{\partial}$ -operator on  $\mathbb{R} \times [0, 1]$  with  $\mathbb{R}^n$  boundary conditions acting on a Sobolev space with weight  $e^{-|\tau|}$ ,  $\tau + it \in \mathbb{R} \times [0, 1]$  is invertible on the complement of cut off solutions, we conclude that  $v_\lambda^{\text{jun}} \rightarrow 0$  as well. This contradicts (6.13).

The proof in case **(gd2)** is similar but simpler.  $\square$

**6.5.3. Existence and uniqueness of solutions near generalized disks.** As mentioned above, after trivializing the bundle of complex anti-linear maps over a neighborhood of  $w_\lambda$ , see Subsection 6.1.6, we consider the  $\bar{\partial}_J$ -operator on maps in a neighborhood of  $w_\lambda$  as a map

$$f: \hat{\mathcal{H}}_{2,\delta} \rightarrow \mathcal{H}_{1,\delta},$$

with  $w_\delta$  corresponding to  $0 \in \hat{\mathcal{H}}_{2,\delta}$ .

**Lemma 6.22.** *There exists a constant  $C > 0$  such that*

$$f(v) = f(0) + df(v) + N(v),$$

*where*

$$\|N(v_1) - N(v_2)\|_{1,\delta} \leq C(\|v_1\|_{2,\delta} + \|v_2\|_{2,\delta})\|v_1 - v_2\|_{2,\delta}.$$

*Proof.* This is a standard argument in the case there are no weights and no cut-off solutions. However, as the weight function is  $\geq 1$ , the left hand side is linear in the weight and the right hand side is quadratic, so the weight does not interfere with the estimate. The cut-off solutions are true solutions except in finite regions where the weight is bounded. We conclude that the estimate holds.  $\square$

**Corollary 6.23.** *There exists a unique disk in a finite  $\|\cdot\|_{2,\delta}$ -neighborhood of  $w_\lambda$  for all sufficiently small  $\lambda > 0$ .*

*Proof.* This is a consequence of Lemma 6.22 in combination with Lemma 6.2.  $\square$

**Lemma 6.24.** *For sufficiently small  $\lambda > 0$ , if a holomorphic disk lies in a sufficiently small  $C^0$ -neighborhood of  $w_\lambda$ , then it lies inside a small  $\|\cdot\|_{2,\delta}$ -neighborhood of  $w_\lambda$ .*

*Proof.* The proof of this lemma is analogous to the last argument in the proof of [3, Theorem 1.3], so we only sketch it. Fix a generalized disk. It is a consequence of the convergence result Corollary 6.19 that for sufficiently small  $\lambda$ , any holomorphic disk in a finite neighborhood of the generalized disk converges to it as  $\lambda \rightarrow 0$ . Further, the domains of such a sequence of disks can be subdivided into three pieces: a subset of a domain  $\Delta_m(\lambda)$  which converges to the domain  $\Delta_m$  of the limit disk on which the map converges to the limit map, a strip part of length  $O(\lambda^{-1})$  mapping to a small neighborhood of the junction point, and a half infinite strip converging to a Morse flow line. We estimate the  $\|\cdot\|_{2,\delta}$ -distance by considering these three pieces separately. Over the big-disk part, closeness to  $w_\lambda$  in the  $\|\cdot\|_{2,\delta}$ -norm follows from the convergence result just mentioned using the decomposition of holomorphic functions into cut-off constant solutions and exponentially decaying functions near the marked point puncture. Over the gradient line part, the  $\|\cdot\|_{2,\delta}$ -norm can be controlled exactly as in [3, Proof of Theorem 1.3] (cf. the calculation on page 1218 and the argument for the “finite number of strip regions” on page 1219). Finally, in the region over the junction point we note that if  $y_\lambda$  is any solution and if  $u_\lambda^{\text{jun}}$  is the local solution that was used to build  $w_\lambda$ , then  $y_\lambda - u_\lambda^{\text{jun}}$  maps into  $\mathbb{C}^n$ , is holomorphic, and satisfies  $\mathbb{R}^n$  boundary conditions. As in the proof of [3, Theorem 1.3], we find that the  $C^0$ -distance near the endpoints of the strip region controls the  $\|\cdot\|_{2,\delta}$ -norm and the  $C^0$ -distance goes to 0 by Corollary 6.19.  $\square$

**Corollary 6.25.** *For all sufficiently small  $\lambda > 0$ , there is a unique rigid holomorphic disk with at most two Morse chords corresponding to each rigid generalized disk.*

*Proof.* This follows from Corollary 6.23 and Lemma 6.24.  $\square$

**6.6. Proof of Theorem 3.6.** Lemma 6.11 and Corollaries 6.19 and 6.25 imply that there is a family of normalized triples  $(f_\lambda, g, J)$ ,  $0 < \lambda < 1$  such that any rigid holomorphic disk with boundary on  $L \cup L_\lambda$  which has at least one Morse puncture is transversely cut out and corresponds either to a flow line or to a generalized disk. In particular, it follows that (i) and (ii) hold, that (iv) holds for any moduli space of disks with at least one Morse puncture, and that (3) and (4) hold.

It thus remains to deal with disks with two mixed punctures, neither of which are Morse punctures. Consider a mixed puncture of such a disk. At such a puncture, the disk has an incoming and an outgoing sheet. Identify these sheets of  $L \cup L_\lambda$  with the corresponding sheets of  $L$ . This gives asymptotic data for a holomorphic disk with boundary on  $L$  at the Reeb chord corresponding to the mixed Reeb chord we started with. That asymptotic data may correspond to a positive or a negative puncture. We call it the *induced asymptotic data* at the mixed puncture. There are the following two cases to consider.

- (I) The asymptotic data of one of the mixed punctures corresponds to a positive puncture and that of the other corresponds to a negative puncture.

(II) The asymptotic data of both mixed punctures correspond to positive punctures.

To get the correspondence between rigid disks of type (I) and liftings of rigid disks in  $\mathcal{M}(a; b_1, \dots, b_k)$ , we argue as follows. View the boundary condition for a disk of type (I) as a perturbation of the boundary condition for a disk in  $\mathcal{M}(a; b_1, \dots, b_k)$ . As the latter moduli space is transversely cut out, it follows by Gromov compactness that for  $\lambda > 0$  small enough there is a bijective correspondence between rigid disks of type (I) and liftings of rigid disks in  $\mathcal{M}(a; b_1, \dots, b_k)$ .

Fix a small  $\lambda > 0$  so that this bijective correspondence exists and so that Corollary 6.25 holds. Assume that the Morse function  $f_\lambda$  is sufficiently generic so that Lemma 2.5 holds for  $L_0 = L$  and  $L_1 = L_1(f_\lambda) = L_\lambda$ : as the proof of Lemma 2.5 shows, to achieve this genericity we need only perturb  $f_\lambda$  an arbitrary small amount near the double points of  $L$ . For sufficiently small such perturbation  $(f_\lambda, g, J)$  remains normalized. Then, by definition, rigid holomorphic disks of type (II) correspond to liftings of rigid disks in  $\mathcal{M}(a_1, a_2; b_1, \dots, b_k)$ . This shows that (iii), the general version of (iv), as well as (2) hold. As mentioned in the beginning of Subsection 6.4, (1) holds provided  $\lambda > 0$  is small enough, and we conclude that the theorem holds.

#### APPENDIX A. TORSION FRAMINGS AND THE GRADING

The goal of this section is to show how a choice of sections of  $TP$  over the 3-skeleton of some fixed triangulation of  $P$  determines a loop  $\mathbb{Z}_g$ -framing used for grading in Section 2.2.2. We will prove these results in the slightly more general setting of a rank  $k$  complex vector bundle  $\eta \rightarrow M$ , where  $k \geq 2$ . See Section 2.2.2 for the definitions of the greatest divisor  $g(\eta)$  and a  $\mathbb{Z}_g$ -framing of  $\eta$  along a closed curve  $\gamma \subset M$ .

Fix a triangulation  $T$  of  $M$  and let  $T^{(j)}$  denote the  $j$ -skeleton of  $T$ . Fix a Hermitian metric on  $\eta$ .

**Lemma A.1.** *The complex vector bundle  $\eta$  admits  $k - 1$  everywhere orthonormal sections over  $T^{(3)}$ . Any two restrictions of such  $(k - 1)$ -frames over  $T^{(2)}$  are homotopic.*

*Proof.* The first statement follows from the fact that the higher Chern classes of  $\eta|_{T^{(3)}}$  vanish as  $H^i(T^{(3)}) = 0$  for  $i \geq 4$ .

We next consider the uniqueness of such frames. Let  $V(r, r - 1)$  denote the Stiefel manifold of orthonormal  $(r - 1)$ -frames in  $\mathbb{C}^r$  and consider the natural fibration

$$V(r - 1, r - 2) \xrightarrow{\iota} V(r, r - 1) \xrightarrow{\pi} S^{2r-1},$$

where  $\pi$  is the projection which maps a frame to its first vector and where  $\iota$  is the inclusion of the fiber. The long exact homotopy sequences of these fibrations show that for  $j = 1, 2$  we have

$$\pi_j(V(r, r - 1)) \cong \pi_j(V(r - 1, r - 2)) \cong \dots \cong \pi_j(V(2, 1)) = 0, \quad j = 1, 2.$$

A  $(k - 1)$ -frame in  $\eta$  over  $T^{(2)}$  gives a section in the bundle  $V(\eta, k - 1)|_{T^{(2)}}$ , where  $V(\eta, k - 1)$  is the bundle with fiber  $V(k, k - 1)$  naturally associated to  $\eta$ . The obstructions to finding a homotopy between two such sections over  $T^{(j)}$  lie in

$$H^j(T^{(2)}; \pi_j(V(k, k - 1))) = 0, \quad j = 1, 2$$

and the lemma follows.  $\square$

Pick  $k - 1$  orthonormal vector fields  $(v_1, \dots, v_{k-1})$  of  $\eta$  over  $T^{(3)}$ . This induces a decomposition

$$\eta|_{T^{(3)}} = \epsilon_1 \oplus \dots \oplus \epsilon_{k-1} \oplus L,$$

where  $\epsilon_j$ ,  $j = 1, \dots, k - 1$  are trivial and trivialized line bundles and where  $L$  is a line bundle. The line bundle  $L$  has Chern class  $c_1(L) = c_1(\eta) = g(\eta)a$  for some  $g(\eta) \geq 0$  and some  $a \in H^2(M; \mathbb{Z}) = H^2(T^{(3)}; \mathbb{Z})$ . Let  $K$  be a line bundle over  $T^{(3)}$  with  $c_1(K) = a$  and let  $w$  be a generic section of  $K$ . (Here we say that  $w$  is generic if it does not vanish along  $T^{(1)}$  and if its 0-set is transverse to  $T^{(2)}$ .) Since two line bundles are isomorphic if and only if they have the same Chern class, it follows that  $L \cong K^{\otimes g(\eta)}$ . In particular,  $(v_1, \dots, v_{k-1}, w^{g(\eta)})$  gives a framing  $\zeta^{(1)}$  of  $\eta$  over  $T^{(1)}$ .

**Lemma A.2.** *If  $\delta_j: S^1 \rightarrow T^{(1)}$ ,  $j = 1, \dots, m$  are curves in  $T^{(1)}$  and if  $C: \Sigma \rightarrow T^{(2)}$  is any map of an orientable surface with  $m$  boundary components  $\partial\Sigma = \partial\Sigma_1 \cup \dots \cup \partial\Sigma_m$  such that  $C|\partial\Sigma_j = \delta_j$ , then the obstruction to extending the trivialization  $C^*(\zeta^{(1)})$  of  $C^*(\eta)$  from  $\partial\Sigma$  to  $\Sigma$  is a class  $b \in H^2(\Sigma, \partial\Sigma; \mathbb{Z})$  which is divisible by  $g(\eta)$ .*

*Proof.* The obstruction to finding such a trivialization is equal to the obstruction to extending the section  $w^{g(\eta)}$  of  $C^*(L)$  over  $\Sigma$ , which equals  $g(\eta)$  times the obstruction of extending  $w$ .  $\square$

The first cohomology group  $H^1(M; \mathbb{Z}_g)$  of  $M$  acts naturally on  $\mathbb{Z}_g$ -framings as follows. Let  $g \geq 0$  and let  $\gamma$  be a closed curve in  $M$ . Assume that  $\gamma$  is equipped with a  $\mathbb{Z}_g$ -framing  $\Xi_\gamma$  of the complex vector bundle  $\eta$ . If  $b \in H^1(M; \mathbb{Z}_g)$ , then we define an action of  $b$  on the  $\mathbb{Z}_g$ -framing of  $\gamma$ ,  $\Xi_\gamma \mapsto \Xi'_\gamma$  as follows. Pick a framing  $Z_\gamma$  of  $\eta|_\gamma$  representing  $\Xi_\gamma$ . Then the  $\mathbb{Z}_g$ -framing  $\Xi'_\gamma$  is represented by any framing  $Z'_\gamma$  with  $d(Z_\gamma, Z'_\gamma) = \langle b, [\gamma] \rangle \pmod{g}$ , where  $\langle a, \beta \rangle$  denotes the homomorphism  $H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}_g$  corresponding to the cohomology class  $a$  evaluated on the homology class  $\beta$ . Note that in the special case  $g = 0$ , a  $\mathbb{Z}_g$ -framing is simply a framing and  $H^1(M; \mathbb{Z})$  acts on framings.

**Lemma A.3.** *Let  $M$  be a manifold with a complex vector bundle  $\eta$ . Let  $g = g(\eta)$  be the greatest divisor of  $\eta$ , with  $c_1(\eta) = g \cdot a$ . Let  $\gamma$  be any closed curve in  $M$ . Then there is an induced  $\mathbb{Z}_g$ -framing  $\Xi_\gamma$  of  $\gamma$  which is unique up to the action of  $H^1(M; \mathbb{Z}_g)$ . In particular, the  $\mathbb{Z}_g$ -framing of any  $\gamma$  which represents a homology class in  $H_1(M; \mathbb{Z})$  which generates a subgroup isomorphic to  $\mathbb{Z}_m$  where  $m$  and  $g$  are relatively prime is unique. In the special case  $g = 0$ , the framing of any curve representing a torsion class is unique.*

*Proof.* Fix a triangulation  $T$  of  $M$ . Construct a framing  $\zeta^{(1)}$  over  $T^{(1)}$  as described above. Pick a homotopy  $C$  which connects  $\gamma$  to a curve in  $T^{(1)}$ . Transporting the trivialization  $\zeta^{(1)}$  over  $C$  gives a trivialization  $Z_\gamma$  over  $\gamma$ . Let  $C'$  be another homotopy inducing another trivialization  $Z'_\gamma$ . Let  $C''$  be a homotopy in  $T^{(2)}$  connecting the end-curve of  $C$  to the end-curve of  $C'$ . Then the cylinder  $D$  constructed by joining  $C$  to  $C''$  and  $C''$  to  $C'$  connects  $\gamma$  to itself. The obstruction to finding a framing over the torus corresponding to  $D$  is on the one hand equal to  $\langle c_1(\eta), [D] \rangle = g\langle a, [D] \rangle$ . On the other hand, this obstruction equals the sum of  $d(Z_\gamma, Z'_\gamma)$  and the obstruction  $o \in \mathbb{Z}$  to extending the framing  $\zeta^{(1)}$  over  $C''$ . The obstruction  $o$  is divisible by  $g$  by Lemma A.2,  $o = go'$ . Thus

$$d(Z_\gamma, Z'_\gamma) = g(\langle a, [D] \rangle - o'),$$

and existence of  $\Xi_\gamma$  follows.

We next consider uniqueness. Consider applying the above construction in two ways for a fixed triangulation  $T$ . This gives two generic frames  $(v_1, \dots, v_{k-1}, w)$  and  $(v'_1, \dots, v'_{k-1}, w')$  over  $T^{(2)}$ . Lemma A.1 implies that there is a homotopy of the  $(k-1)$ -frames. Note that such a homotopy induces 1-parameter family of bundle isomorphisms on orthonormal complements of the  $(k-1)$ -frame. Deforming  $(v'_1, \dots, v'_{k-1})$  to  $(v_1, \dots, v_{k-1})$  thus gives a new generic frame  $(v_1, \dots, v_{k-1}, w'')$ . The generic frames give the same  $\mathbb{Z}_g$ -framing of all loops in  $T^{(1)}$  provided the corresponding difference classes of the sections  $w$  and  $w''$  of the line bundle  $L$  are divisible by  $g$ . It follows that  $H^1(T^{(2)}; \mathbb{Z}_g)$  acts transitively on homotopy classes of  $\mathbb{Z}_g$ -framings of loops in  $T^{(1)}$ . Thus all loop  $\mathbb{Z}_g$ -framings obtained from a fixed triangulation form a principal homogeneous space over  $H^1(T^{(2)}; \mathbb{Z}_g) = H^1(M; \mathbb{Z}_g)$ .

Let  $S$  be some other triangulation of  $M$ . After small perturbation of  $S$  we may assume that the triangulations  $S$  and  $T$  have a common refinement  $U$ . Using  $U$  we find that there exist trivializations  $\zeta_T^{(1)}$  and  $\zeta_S^{(1)}$  over the 1-skeleta of  $T$  and  $S$ , respectively such that the corresponding loop  $\mathbb{Z}_g$ -framings agree. Since the action of  $H^1(M; \mathbb{Z}_g)$  on  $\mathbb{Z}_g$ -framings is independent of triangulation it follows that the set of loop  $\mathbb{Z}_g$ -framings is independent of the chosen triangulation.  $\square$

*Remark A.4.* Let  $M$  be an orientable manifold. Consider the tangent bundle of the cotangent bundle  $T(T^*M)$ . After fixing an almost complex structure  $J$  on  $T(T^*M)$  which is compatible with the standard symplectic form, we consider  $T(T^*M)$  as complex vector bundle. Since  $T^*M \simeq M$  and since the restriction of  $T(T^*M)$  to  $M$  is the complexification of a real vector bundle, we see that  $c_1(T(T^*M)) = 0$ . Furthermore, an orientation of  $M$  gives a non-zero section of  $\Lambda^{\max} TM$ , which gives a non-zero section of  $\Lambda^{\max} T(T^*M)$ . This, in turn, induces a trivialization over the 3-skeleton of  $M$  following the construction above by requiring that the section  $v_1 \wedge \dots \wedge v_{n-1} \wedge w$ , where  $v_j$  and  $w$  give the trivialization  $\zeta^{(1)}$  in Lemma A.2, is homotopic to the one induced by the orientation. The result is the canonical trivialization discussed after Definition 2.1.

## REFERENCES

- [1] M. Betz and R. L. Cohen, *Graph moduli spaces and cohomology operations*, Turkish J. Math. **18** (1994), no. 1, 23–41.
- [2] Yu. Chekanov, *Differential algebra of Legendrian links*, Invent. Math. **150** (2002), 441–483.
- [3] T. Ekholm, *Morse flow trees and Legendrian contact homology in 1-jet spaces*, Geom. Topol. **11** (2007), 1083–1224.
- [4] T. Ekholm, *Rational symplectic field theory over  $\mathbb{Z}_2$  for exact Lagrangian cobordisms*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 3, 641–704.
- [5] T. Ekholm, J. Etnyre, and M. Sullivan, *The contact homology of Legendrian submanifolds in  $\mathbb{R}^{2n+1}$* , J. Differential Geom. **71** (2005), no. 2, 177–305.
- [6] ———, *Non-isotopic Legendrian submanifolds in  $\mathbb{R}^{2n+1}$* , J. Differential Geom. **71** (2005), no. 1, 85–128.
- [7] ———, *Orientations in Legendrian contact homology and exact Lagrangian immersions*, Internat. J. Math. **16** (2005), no. 5, 453–532.
- [8] ———, *Legendrian contact homology in  $P \times \mathbb{R}$* , Trans. Amer. Math. Soc. **359** (2007), no. 7, 3301–3335 (electronic).
- [9] Ya. Eliashberg, *Invariants in contact topology*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 327–338 (electronic).

- [10] Ya. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. (2000), no. Special Volume, Part II, 560–673.
- [11] J. Epstein and D. Fuchs, *On the invariants of Legendrian mirror torus links*, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 103–115.
- [12] A. Floer, *Monopoles on asymptotically flat manifolds (3–41)*, in The Floer memorial volume. Edited by Helmut Hofer, Clifford H. Taubes, Alan Weinstein and Eduard Zehnder. Progress in Mathematics, 133. Birkhäuser Verlag, Basel, 1995.
- [13] D. Fuchs, *Chekanov-Eliashberg invariant of Legendrian knots: existence of augmentations*, J. Geom. Phys. **47** (2003), no. 1, 43–65.
- [14] D. Fuchs and T. Ishkhanov, *Invariants of Legendrian knots and decompositions of front diagrams*, Mosc. Math. J. **4** (2004), no. 3, 707–717.
- [15] K. Fukaya, Y.-G. Oh, T. Ohta, and K. Ono, *Lagrangian intersection Floer homology — anomaly and obstruction*, Preprint, 2000.
- [16] P. Melvin and S. Shrestha, *The nonuniqueness of Chekanov polynomials of Legendrian knots*, Geom. Topol. **9** (2005), 1221–1252.
- [17] D. Milinković, *Morse homology for generating functions of Lagrangian submanifolds*, Trans. Amer. Math. Soc. **351** (1999), no. 10, 3953–3974.
- [18] K. Mishachev, *The  $n$ -copy of a topologically trivial Legendrian knot*, J. Symplectic Geom. **1** (2003), no. 4, 659–682.
- [19] L. Ng, *Computable Legendrian invariants*, Topology **42** (2003), no. 1, 55–82.
- [20] ———, *Framed knot contact homology*, Preprint available on arXiv as math.GT/0407071, 2004.
- [21] L. Ng and J. Sabloff, *The correspondence between augmentations and rulings for Legendrian knots*, Pacific J. Math. **224** (2006), no. 1, 141–150.
- [22] L. Ng and L. Traynor, *Legendrian solid-torus links*, J. Symplectic Geom. **2** (2004), no. 3, 411–443.
- [23] J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993), 827–844.
- [24] D. Rutherford, *Thurston-Bennequin number, Kauffman polynomial, and ruling invariants of a Legendrian link: the Fuchs conjecture and beyond*, Int. Math. Res. Not. (2006), Art. ID 78591, 15.
- [25] J. Sabloff, *Augmentations and rulings of Legendrian knots*, Int. Math. Res. Not. (2005), no. 19, 1157–1180.
- [26] ———, *Duality for Legendrian contact homology*, Geom. Topol. **10** (2006), 2351–2381 (electronic).
- [27] M. Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993.
- [28] ———, *Equivalences for Morse homology*, Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999), Contemp. Math., vol. 246, Amer. Math. Soc., Providence, RI, 1999, pp. 197–216.
- [29] S. Smale, *On gradient dynamical systems*, Ann. of Math. (2) **74** (1961), 199–206.

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